MAT320/640, Fall 2021 **Renato Ghini Bettiol**

Homework Set 4

DUE: OCT 25, 2021 (VIA BLACKBOARD, BY 11.59PM)

To be handed in:

Please remember that all problems will be graded!

1. Use one of the convergence tests we discussed during lectures to justify whether each of the following series *converges* (and, if so, *absolutely*?) or *diverges*.

   (a) $\displaystyle\sum_{n=1}^{\infty} 3n^2\, e^{-n^3}$

   (b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\,\frac{2^n}{n^2}$

   (c) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\,\frac{n^2}{2^n}$

   (d) $\displaystyle\sum_{n=1}^{\infty}\left[\sin\left(\frac{n\pi}{7}\right)\right]^n$

   (e) $\displaystyle\sum_{n=1}^{\infty}\left[\sin\left(\frac{n\pi}{8}\right)\right]^n$

   (f) $\displaystyle\frac{1}{\pi}+\frac{1}{5}+\frac{1}{\pi^2}+\frac{1}{5^2}+\frac{1}{\pi^3}+\frac{1}{5^3}+\frac{1}{\pi^4}+\frac{1}{5^4}+\dots$

Solutions

Text in blue represents side comments that are not integral parts of proofs, but address issues that some students might have had difficulties with in their attempted solutions.

(a) $\displaystyle\sum_{n=1}^{\infty} 3n^2\, e^{-n^3} < +\infty$ **converges absolutely**

Let $f\colon [1,+\infty)\to\mathbb{R}$ be given by $f(x)=3x^2e^{-x^3}$. Then $f(x)$ is continuous, $f(x)>0$, and decreasing, since $f'(x)=(6x-9x^4)e^{-x^3}<0$ for all $x\geq 1$. Moreover,

$$\int_1^{+\infty} f(x)\,\mathrm{d}x = \lim_{b\to+\infty}\int_1^b f(x)\,\mathrm{d}x = \lim_{b\to+\infty} -e^{-b^3}+e^{-1} = \frac{1}{e} < +\infty,$$

so the series $\sum_{n=1}^{\infty} f(n)$ converges, by the Integral Test (Lecture 11). Moreover, the series converges *absolutely* since $a_n\geq 0$, so $\sum |a_n| = \sum a_n$.

Another possible solution: use the Ratio Test.

(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n^2}$ **diverges**

The sequence $|a_n| = \left|(-1)^{n+1}\frac{2^n}{n^2}\right| = \frac{2^n}{n^2}$ diverges to $+\infty$ as $n \to +\infty$, therefore the series $\sum_{n=1}^{+\infty} a_n$ diverges by the "n-th term test", since $\lim_{n\to+\infty} a_n = 0$ is a necessary condition for convergence of the series (Lecture 9).

(c) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{2^n}$ **converges absolutely**

The sequence $a_n = (-1)^{n+1}\frac{n^2}{2^n}$ is such that $|a_n|^{1/n} = \frac{n^{2/n}}{2}$, so

$$\limsup_{n\to+\infty} |a_n|^{1/n} = \lim_{n\to+\infty} \frac{n^{2/n}}{2} = \frac{1}{2} < 1,$$

so the series $\sum a_n$ converges absolutely by the Root Test.

Another possible (but more complicated) solution:
The sequence $a_n = \frac{n^2}{2^n}$ satisfies $a_n \geq 0$, is decreasing (for $n \geq 3$), and $\lim_{n\to+\infty} a_n = 0$. Thus, $\sum_{n=1}^{+\infty}(-1)^{n+1}a_n$ converges by the Alternating Series Test (Lecture 11). Moreover, the above series converges *absolutely* by the Integral Test, since the function $f\colon [1,+\infty) \to \mathbb{R}$ given by $f(x) = \frac{x^2}{2^x}$ is continuous, $f(x) > 0$, and decreasing (for $x \geq \frac{2}{\ln 2}$), because $f'(x) = -2^x x(x\ln 2 - 2) < 0$ if $x \geq \frac{2}{\ln 2} \cong 2.89$, and $\int_1^{+\infty} f(x)\,\mathrm{d}x = \frac{\ln 4 + \ln^2 2 + 2}{2\ln^3 2} < +\infty$, so $\sum_{n=1}^{+\infty} f(n) < +\infty$ by the Integral Test (Lecture 11).

(d) $\displaystyle\sum_{n=1}^{\infty}\left[\sin\left(\frac{n\pi}{7}\right)\right]^n$ **converges absolutely**

The sequence $a_n = \left[\sin\left(\frac{n\pi}{7}\right)\right]^n$ is such that $|a_n|^{1/n} = \left|\sin\left(\frac{n\pi}{7}\right)\right|$, which is periodic, and repeats itself each time n grows by 7; namely, $|a_{n+7}|^{1/(n+7)} = |a_n|^{1/n}$ for all $n \in \mathbb{N}$. Thus, the list of all possible values that $|a_n|^{1/n}$ assumes is the following: 0, $\sin\left(\frac{\pi}{7}\right)$, $\cos\left(\frac{3\pi}{14}\right)$, $\cos\left(\frac{\pi}{14}\right)$; and it follows that the largest subsequential limit is

$$\limsup_{n\to+\infty} |a_n|^{1/n} = \cos\left(\frac{\pi}{14}\right) \cong 0.97 < 1.$$

Thus, $\sum_{n=1}^{+\infty} a_n$ converges absolutely by the Root Test (Lecture 9).

(e) $\displaystyle\sum_{n=1}^{\infty}\left[\sin\left(\frac{n\pi}{8}\right)\right]^n$ **diverges**

The sequence $a_n = \left[\sin\left(\frac{n\pi}{8}\right)\right]^n$ is such that $|a_n|^{1/n} = \left|\sin\left(\frac{n\pi}{8}\right)\right|$, which is periodic, and repeats itself each time n grows by 8; namely, $|a_{n+8}|^{1/(n+8)} = |a_n|^{1/n}$ for all

$n \in \mathbb{N}$. Thus, the list of all possible values that $|a_n|^{1/n}$ assumes is the following: $\sin\left(\frac{\pi}{8}\right)$, $\frac{1}{\sqrt{2}}$, $\cos\left(\frac{\pi}{8}\right)$, 1; and it follows that the largest subsequential limit is:

$$\limsup_{n\to+\infty} |a_n|^{1/n} = 1.$$

Even though the Root Test is *inconclusive* in this case, by the above analysis, we can infer that $a_n = 1$ for *infinitely many values* of $n \in \mathbb{N}$, namely all integers of the form $n = 16k+4$ for some $k \in \mathbb{N}$. Therefore, a_n does not converge to 0 as $n \to +\infty$, so $\sum_{n=1}^{+\infty} a_n$ diverges by the "n-th term test" (Lecture 9).

(f) $\dfrac{1}{\pi} + \dfrac{1}{5} + \dfrac{1}{\pi^2} + \dfrac{1}{5^2} + \dfrac{1}{\pi^3} + \dfrac{1}{5^3} + \dfrac{1}{\pi^4} + \dfrac{1}{5^4} + \dots$ **converges absolutely**

By an analysis similar to the last exercise in Lecture 11, we have that the coefficients a_n in the above series $\sum_{n=1}^{+\infty} a_n$ satisfy:

$$|a_n|^{1/n} = \begin{cases} \left(\dfrac{1}{\pi^m}\right)^{\frac{1}{2m-1}} & \text{if } n = 2m-1 \text{ is odd} \\[2ex] \left(\dfrac{1}{5^m}\right)^{\frac{1}{2m}} = \dfrac{1}{\sqrt{5}} & \text{if } n = 2m \text{ is even} \end{cases}$$

Therefore, we compute:

$$\limsup_{n\to+\infty} |a_n|^{1/n} = \lim_{m\to+\infty} \left(\frac{1}{\pi^m}\right)^{\frac{1}{2m-1}} = \frac{1}{\sqrt{\pi}} < 1,$$

so the series converges absolutely by the Root Test.