MAT320/640, Fall 2021 **Renato Ghini Bettiol**

Homework Set 5

Due: Nov 8, 2021 (via Blackboard, by 11.59pm)

To be handed in:

Please remember that all problems will be graded!

1. Give a rigorous proof that $f(x) = \frac{1}{1-x^2}$ is continuous at any $x_0 \in (-1,1)$, explicitly finding $\delta > 0$ for each $\varepsilon > 0$. Does the δ you found depend on x_0 or only on ε?

2. (a) Is the function $f(x) = \frac{1}{1-x^2}$ uniformly continuous on $[-\frac{1}{2}, \frac{1}{2}]$? Justify.

   (b) Is the function $f(x) = \frac{1}{1-x^2}$ uniformly continuous on $(-1,1)$? Justify.

Solutions

Text in blue represents side comments that are not integral parts of proofs, but address issues that some students might have had difficulties with in their attempted solutions.

1. Let $f\colon (-1,1) \to \mathbb{R}$ be given by $f(x) = \frac{1}{1-x^2}$, and fix $x_0 \in (-1,1)$. Note that

$$|f(x) - f(x_0)| = \left| \frac{1}{1-x^2} - \frac{1}{1-x_0^2} \right| = \left| \frac{(1-x_0^2) - (1-x^2)}{(1-x^2)(1-x_0^2)} \right|$$
$$= \frac{|x^2 - x_0^2|}{(1-x^2)(1-x_0^2)} = \frac{|x-x_0|\,|x+x_0|}{(1-x^2)(1-x_0^2)}.$$

Thus, in order to make the above be $< \varepsilon$ when x is such that $|x - x_0| < \delta$, we need to bound $|x + x_0|$, which is easy, and $\frac{1}{1-x^2}$, which is just a bit harder.

For the first, note that $|x+x_0| \leq |x-x_0| + 2|x_0| < \delta + 2|x_0|$ by the triangle inequality. For the second, we first solve $\frac{1}{1-x^2} \leq c$ in x, see equation (1) below, which leads us to consider the following strategy. If $\delta < 1 - |x_0|$, then there exists $c > 0$ such that $(\delta + |x_0|)^2 = 1 - \frac{1}{c}$. More precisely, let $c = \frac{1}{1-(|x_0|+\delta)^2}$. By the triangle inequality, we have $|x| \leq |x - x_0| + |x_0| < \delta + |x_0| = \sqrt{1 - \frac{1}{c}}$ which, in turn, implies $\frac{1}{1-x^2} < c$, as

$$|x| < \sqrt{1 - \frac{1}{c}} \iff x^2 < 1 - \frac{1}{c} \iff 1 < c - cx^2 = c(1-x^2) \iff \frac{1}{1-x^2} < c. \quad (1)$$

So, altogether, given $\varepsilon > 0$, we choose $\delta > 0$ such that $\delta < 1 - |x_0|$, and

$$\frac{\delta(\delta + 2|x_0|)}{(1-x_0^2)(1-(|x_0|+\delta)^2)} < \varepsilon.$$

Then, if $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| = \frac{|x-x_0||x+x_0|}{(1-x^2)(1-x_0^2)} < \frac{\delta(\delta+2|x_0|)}{(1-x_0^2)(1-(|x_0|+\delta)^2)} < \varepsilon$. Clearly, the above choice of δ depends on both ε and x_0, and $\delta \searrow 0$ when $|x_0| \nearrow 1$.

2. (a) **Yes.** The function $f(x) = \frac{1}{1-x^2}$ is continuous on $(-1, 1)$ by Problem 1, and thus *uniformly continuous* on any closed subinterval, such as $[-\frac{1}{2}, \frac{1}{2}] \subset (-1, 1)$, see Lecture 14 (video 3).

   (b) **No.** If the function $f(x) = \frac{1}{1-x^2}$ was uniformly continuous on $(-1, 1)$, then there would be a continuous extension $\widetilde{f}\colon [-1, 1] \to \mathbb{R}$ of $f(x)$, see Lecture 14 (video 5), but this is impossible since $\lim\limits_{x \to 1_-} f(x) = \lim\limits_{x \to -1_+} f(x) = +\infty$.