Homework Set 6

DUE: NOV 22, 2021 (VIA BLACKBOARD, BY 11.59PM)

To be handed in:

Please remember that all problems will be graded!

1. Consider the sequence of functions $f_n(x) = \dfrac{n+\cos x}{3n+\sin^2 x}$ for all $x \in \mathbb{R}$.
   (a) Find an explicit function $f\colon \mathbb{R} \to \mathbb{R}$ such that $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $\mathbb{R}$ to f. You must justify why the convergence is uniform by verifying the definition of uniform convergence.
   (b) Use the function $f(x)$ to compute $\displaystyle\lim_{n\to+\infty} \int_1^5 f_n(x)\,\mathrm{d}x$.

2. (a) Use differentiation term-by-term and an example from class (Lectures 15/18) to prove that $\displaystyle\sum_{n=1}^{+\infty} nx^n = \frac{x}{(1-x)^2}$ for all $|x| < 1$.
   (b) Compute $\displaystyle\sum_{n=1}^{+\infty} \frac{n}{2^n}$.

Solutions

Text in blue represents side comments that are not integral parts of proofs, but address issues that some students might have had difficulties with in their attempted solutions.

1. (a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be the constant function given by $f(x) = \frac{1}{3}$. In order to show that $f_n(x)$ converge uniformly to $f(x)$, we must prove that, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$ and $x \in \mathbb{R}$. So, we compute:

$$|f_n(x) - f(x)| = \left|\frac{n+\cos x}{3n+\sin^2 x} - \frac{1}{3}\right| = \left|\frac{3(n+\cos x) - (3n+\sin^2 x)}{3(3n+\sin^2 x)}\right| =$$
$$= \left|\frac{3\cos x - \sin^2 x}{3(3n+\sin^2 x)}\right| \leq \frac{\left|3\cos x - \sin^2 x\right|}{9n} \leq \frac{4}{9n},$$

where the first inequality follows from $3n + \sin^2 x \geq 3n$ for all $x \in \mathbb{R}$, and the second inequality follows from the triangle inequality:

$$\left|3\cos x - \sin^2 x\right| \leq 3\,|\cos x| + \left|\sin^2 x\right| \leq 3 + 1 = 4.$$

Therefore, if we take $N \in \mathbb{N}$ to be the smallest integer larger than $\frac{4}{9\varepsilon}$, then for all $n \geq N > \frac{4}{9\varepsilon}$ it follows from the above that $|f_n(x) - f(x)| \leq \frac{4}{9n} < \varepsilon$, as desired. $\square$

(b) Since $f_n(x)$ are continuous for all $n \in \mathbb{N}$ and converge uniformly to $f(x) = \frac{1}{3}$, we may exchange the order of limit and integration (see Video 1 of Lecture 17):

$$\lim_{n\to+\infty} \int_1^5 f_n(x)\,\mathrm{d}x = \int_1^5 \lim_{n\to+\infty} f_n(x)\,\mathrm{d}x = \int_1^5 \tfrac{1}{3}\,\mathrm{d}x = \tfrac{1}{3}(5-1) = \tfrac{4}{3}.$$

2. (a) From Lecture 18, the following power series has radius of convergence $R = 1$:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots, \qquad \text{if } |x| < 1.$$

Thus, differentiating term-by-term, we have that:

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1-x} = \frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}x^n = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + \dots, \qquad \text{if } |x| < 1.$$

On the other hand, by the Chain Rule, we find $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1-x} = \frac{1}{(1-x)^2}$, so, replacing this in the above and multiplying both sides by x, we conclude:

$$\frac{x}{(1-x)^2} = x\sum_{n=1}^{+\infty} n x^{n-1} = \sum_{n=1}^{+\infty} n x^n, \qquad \text{if } |x| < 1.$$

(b) Using $x = \frac{1}{2}$ in the above, we find $\displaystyle\sum_{n=1}^{+\infty} \frac{n}{2^n} = \frac{\frac{1}{2}}{\left(1 - \frac{1}{2}\right)^2} = 2.$