MAT320/640, Fall 2021 **Renato Ghini Bettiol**

Homework Set 7

DUE: DEC 6, 2021 (VIA BLACKBOARD, BY 11.59PM)

To be handed in:

Please remember that all problems will be graded!

1. Prove that $|\ln x - \ln y| \leq 5|x-y|$ for all $x, y \in \left[\frac{1}{5}, 5\right]$.

2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a smooth function such that $f(0) = 3$ and all derivatives of $f(x)$ vanish at $x = 0$, that is, $f'(0) = f''(0) = \cdots = f^{(n)}(0) = \cdots = 0$ for all $n \in \mathbb{N}$. Does there exist $\varepsilon > 0$ such that $f(x) = 3$ for all $x \in (-\varepsilon, \varepsilon)$?

3. Consider the function $f\colon [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}$$

   (a) Compute explicitly the lower and upper (Darboux) integrals of $f(x)$ on the interval $[0,1]$, that is, find the values of $U(f)$ and $L(f)$.

   (b) Is $f(x)$ integrable on $[0,1]$?

Solutions

Text in blue represents side comments that are not integral parts of proofs, but address issues that some students might have had difficulties with in their attempted solutions.

1. The function $f\colon \left[\frac{1}{5}, 5\right] \to \mathbb{R}$ given by $f(t) = \ln t$ is continuous on $\left[\frac{1}{5}, 5\right]$ and differentiable on $\left(\frac{1}{5}, 5\right)$, with $f'(t) = \frac{1}{t}$. Thus, by the Mean Value Theorem (Lecture 20), given any $x, y \in \left[\frac{1}{5}, 5\right]$, say with $y < x$, there exists $t_0 \in (y, x)$ such that

$$\ln x - \ln y = \frac{1}{t_0}(x-y).$$

   Since $t_0 \in \left[\frac{1}{5}, 5\right]$, we have $\left|\frac{1}{t_0}\right| \leq 5$, and thus

$$|\ln x - \ln y| = \left|\frac{1}{t_0}\right| |x-y| \leq 5|x-y|.$$

2. **No.** For example, consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by (cf. Lecture 21, Video 4):

$$f(x) = \begin{cases} 3 + e^{-1/x}, & \text{if } x > 0 \\ 3, & \text{if } x \leq 0. \end{cases}$$

Clearly, there does not exist $\varepsilon > 0$ such that $f(x) = 3$ for all $x \in (-\varepsilon, \varepsilon)$, since $f(x) > 3$ for all $x > 0$. Moreover, every derivative of $f(x)$ vanishes at $x = 0$, i.e., $f'(0) = f''(0) = \cdots = f^{(n)}(0) = \cdots = 0$ for all $n \in \mathbb{N}$, as seen in Lecture 21.

3. (a) Let $P = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ be any partition of $[0,1]$. Since both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$, for every $k = 1, \dots, n$, we easily compute:

$$\begin{aligned} m(f, [t_{k-1}, t_k]) &= \inf\{f(x) : x \in [t_{k-1}, t_k]\} = 0, \\ M(f, [t_{k-1}, t_k]) &= \sup\{f(x) : x \in [t_{k-1}, t_k]\} = t_k. \end{aligned}$$

Thus, we conclude that $L(f,P) = 0$ and hence the lower integral of f on $[0,1]$ is $L(f) = 0$. Moreover, if we let $g\colon [0,1] \to \mathbb{R}$ be the function $g(x) = x$, we see that

$$M(f, [t_{k-1}, t_k]) = t_k = M(g, [t_{k-1}, t_k]), \quad \text{for all } k = 1, \dots, n,$$

and hence $U(f,P) = U(g,P)$. Recall g is integrable on $[0,1]$ and $L(g) = U(g) = \int_0^1 g = \frac{1}{2}$. Thus, the upper integral of f on $[0,1]$ is

$$\begin{aligned} U(f) &= \inf\{U(f,P) : P \text{ partition of } [0,1]\} \\ &= \inf\{U(g,P) : P \text{ partition of } [0,1]\} \\ &= U(g) = \tfrac{1}{2}. \end{aligned}$$

(b) **No.** On the interval $[0,1]$, we have $L(f) = 0$ and $U(f) = \frac{1}{2}$, hence $L(f) \neq U(f)$ and thus f is not integrable on $[0,1]$.