Problem 1 (10 pts): The largest currently known *Mersenne prime* is $2^{82,589,933} - 1$. Prove that:

$$\sqrt{2^{82,589,933} - 1} \text{ is not a rational number.}$$

Hint: Let $p(x) = x^2 - (2^{82,589,933} - 1)$ *and use the above fact that* $2^{82,589,933} - 1$ *is prime.*

Solution (following HW1 Problem 2):
Consider the polynomial $p(x) = x^2 - (2^{82,589,933} - 1)$, and note that $\sqrt{2^{82,589,933} - 1}$ is, by definition, the only positive root of $p(x)$. Since $p(x)$ is monic, all its coefficients are integers, and its constant coefficient is $a_0 = 2^{82,589,933} - 1$, the Rational Zeros Theorem (Lecture 2) implies that any rational root of $p(x)$ must be among the divisors of $2^{82,589,933} - 1$. As $2^{82,589,933} - 1$ is a prime number, its divisors are ± 1 and $\pm 2^{82,589,933} - 1$. By direct inspection, $p(\pm 1) \neq 0$ and $p(\pm 2^{82,589,933} - 1) \neq 0$, thus $\sqrt{2^{82,589,933} - 1} \notin \mathbb{Q}$.

Problem 2 (10 pts): Let s_n be a sequence of numbers in the closed interval $[-10, 10]$.

(a) Does s_n have to be Cauchy? Justify.

(b) Does s_n have to admit a Cauchy subsequence? Justify.

Hint: "Justify" means "give a proof" if you answer YES, and it means "give a counter-example" if you answer NO.

Solution:

(a) **No.** The fact that s_n is bounded, more precisely $|s_n| \leq 10$, is not enough to ensure that s_n is Cauchy. As a counter-example, take the sequence $s_n = (-1)^n$. Clearly, $|s_n| = 1 < 10$ for all $n \in \mathbb{N}$, but s_n does not converge, so it is not Cauchy.[1]

(b) **Yes.** By the Bolzano–Weierstrass Theorem (Lecture 8), every bounded sequence of real numbers admits a convergent subsequence. Thus, since s_n is bounded, it admits a convergent subsequence, and this subsequence is Cauchy because it converges.

[1] Recall that a sequence of real numbers is Cauchy if and only if it converges.

Problem 3 (15 pts): Recall from Lecture 21 that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad \text{for all } x \in \mathbb{R}.$$

Starting from the above fact, justify every step of the way to prove that:

$$\int_0^1 \frac{e^{-x^2}}{3}\, \mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{3(2n+1)(n!)}.$$

Solution:
First, we perform the substitution $x \mapsto -x^2$, and obtain

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}, \qquad \text{for all } x \in \mathbb{R}.$$

Next, we divide by 3 on both sides and use $(-x^2)^n = ((-1)(x^2))^n = (-1)^n x^{2n}$ to obtain:

$$\frac{e^{-x^2}}{3} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{3\, n!}, \qquad \text{for all } x \in \mathbb{R}.$$

Note that the radius of convergence $R = +\infty$ is *unchanged* by both of these operations. Now, integrating term-by-term (see Lecture 18) from $x = 0$ to $x = 1$ we find:

$$\int_0^1 \frac{e^{-x^2}}{3}\, \mathrm{d}x = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{3\, n!}\, \mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{3} \int_0^1 \frac{x^{2n}}{n!}\, \mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{3(2n+1)(n!)}.$$

Problem 4 (15 pts): For what values of $x \in \mathbb{R}$ is the following series absolutely convergent?

$$\frac{x}{5} + \frac{x^2}{10} + \frac{x^3}{5^2} + \frac{x^4}{10^2} + \frac{x^5}{5^3} + \frac{x^6}{10^3} + \frac{x^7}{5^4} + \frac{x^8}{10^4} + \dots$$

Solution:

By an analysis similar to the last exercise in Lecture 11 and HW4 Problem 1(f), we have that the above series can be written as $\sum_{n=1}^{+\infty} a_n x^n$, where a_n are given by

$$a_n = \begin{cases} \dfrac{1}{5^m} & \text{if } n = 2m-1 \text{ is odd} \\[2ex] \dfrac{1}{10^m} & \text{if } n = 2m \text{ is even} \end{cases}$$

and, hence,

$$|a_n|^{1/n} = \begin{cases} \left(\dfrac{1}{5^m}\right)^{\frac{1}{2m-1}} = \dfrac{1}{5^{m/(2m-1)}} & \text{if } n = 2m-1 \text{ is odd} \\[2ex] \left(\dfrac{1}{10^m}\right)^{\frac{1}{2m}} = \dfrac{1}{\sqrt{10}} & \text{if } n = 2m \text{ is even} \end{cases}$$

Therefore, the largest subsequential limit is clearly

$$\beta = \limsup_{n\to+\infty} |a_n|^{1/n} = \lim_{m\to+\infty} \frac{1}{5^{m/(2m-1)}} = \frac{1}{\sqrt{5}},$$

and so the radius of convergence for the series is $R = \frac{1}{\beta} = \sqrt{5}$.

Regarding the endpoints $x = \pm\sqrt{5}$, we see that the sum of just the odd terms in the above series already diverges if $x = \sqrt{5}$, since

$$\frac{5^{1/2}}{5} + \frac{5^{3/2}}{5^2} + \frac{5^{5/2}}{5^3} + \frac{5^{7/2}}{5^4} + \dots = \sum_{m=1}^{\infty} \frac{5^{(2m-1)/2}}{5^m} = \sum_{m=1}^{\infty} \frac{1}{\sqrt{5}} = +\infty,$$

so the series does not converge absolutely at neither endpoint $x = \pm\sqrt{5}$.

In conclusion, the above series converges absolutely if and only if $|x| < \sqrt{5}$.

Problem 5 (15 pts): Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a function that satisfies

$$|f(x) - f(y)| \leq C\,|x - y|^{\alpha}$$

for all $x, y \in \mathbb{R}$, where $C > 0$ and $\alpha > 0$ are constants.

(a) Prove that f is uniformly continuous on $\mathbb{R}$.

(b) Give an example of $f(x)$ that satisfies the above with $\alpha = 1$ but is not differentiable at $x = 0$.

(c) Prove that if $\alpha > 1$, then f is constant.

Solution:

(a) Let $\varepsilon > 0$ be given, and let $\delta = \left(\frac{\varepsilon}{C}\right)^{1/\alpha}$. If $x, y \in \mathbb{R}$ satisfy $|x - y| < \delta$, then

$$|f(x) - f(y)| \leq C\,|x - y|^{\alpha} < C\,\delta^{\alpha} = \varepsilon,$$

so f is uniformly continuous.

(b) Let $f(x) = |x|$. Then $f(x)$ is not differentiable at $x = 0$; however, by the triangle inequality, $\big|f(x) - f(y)\big| = \big||x| - |y|\big| \leq |x - y|$ for all $x, y \in \mathbb{R}$, so f satisfies the required property with $C = 1$ and $\alpha = 1$.

(c) If $\alpha > 1$, then setting $y = x_0$ and using the above inequality we can compute

$$|f'(x_0)| = \lim_{x \to x_0} \left|\frac{f(x) - f(x_0)}{x - x_0}\right| \leq \lim_{x \to x_0} \frac{C|x - x_0|^{\alpha}}{|x - x_0|} = C \lim_{x \to x_0} |x - x_0|^{\alpha - 1} = 0,$$

so $f'(x_0) = 0$ for all $x_0 \in \mathbb{R}$, which implies that f is constant.

Problem 6 (15 pts): Compute the following limit of definite integrals:

$$\lim_{n\to+\infty}\int_0^{\pi}\frac{n^2-\sin^3 x}{4n^2+\cos^2 x}\,\mathrm{d}x$$

Hint: Show that $f_n(x)=\dfrac{n^2-\sin^3 x}{4n^2+\cos^2 x}$ *converge uniformly to some* $f(x)$ *as* $n\to+\infty$.
You must justify why the convergence is uniform if you later use that fact.

Solution (following HW6 Problem 1):
Let $f\colon\mathbb{R}\to\mathbb{R}$ be the constant function given by $f(x)=\frac{1}{4}$. In order to show that

$$f_n(x)=\frac{n^2-\sin^3 x}{4n^2+\cos^2 x}$$

converge uniformly to $f(x)$, we must prove that, given $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that $|f_n(x)-f(x)|<\varepsilon$ for all $n\geq N$ and $x\in\mathbb{R}$. So, we compute:

$$|f_n(x)-f(x)|=\left|\frac{n^2-\sin^3 x}{4n^2+\cos^2 x}-\frac{1}{4}\right|=\left|\frac{4(n^2-\sin^3 x)-(4n^2+\cos^2 x)}{4(4n^2+\cos^2 x)}\right|=$$
$$=\left|\frac{4\sin^3 x+\cos^2 x}{4(4n^2+\cos^2 x)}\right|\leq\frac{\left|4\sin^3 x+\cos^2 x\right|}{16n^2}\leq\frac{5}{16n^2},$$

where the first inequality follows from $4n^2+\cos^2 x\geq 4n^2$ for all $x\in\mathbb{R}$, and the second inequality follows from the triangle inequality:

$$\left|4\sin^3 x+\cos^2 x\right|\leq 4\left|\sin^3 x\right|+\left|\cos^2 x\right|\leq 4+1=5.$$

Therefore, if we take $N\in\mathbb{N}$ to be the smallest integer larger than $\sqrt{\frac{5}{16\varepsilon}}=\frac{1}{4}\sqrt{\frac{5}{\varepsilon}}$, then for all $n\geq N$ it follows from the above that $|f_n(x)-f(x)|\leq\frac{5}{16n^2}<\varepsilon$, as desired.
Now, since $f_n(x)$ are continuous for all $n\in\mathbb{N}$ and converge uniformly to $f(x)=\frac{1}{4}$, we may exchange the order of limit and integration (see Video 1 of Lecture 17):

$$\lim_{n\to+\infty}\int_0^{\pi}f_n(x)\,\mathrm{d}x=\int_0^{\pi}\lim_{n\to+\infty}f_n(x)\,\mathrm{d}x=\int_0^{\pi}\tfrac{1}{4}\,\mathrm{d}x=\tfrac{\pi}{4}.$$

Problem 7 (20 pts): Consider the sequence of functions $f_n\colon [0,1] \to \mathbb{R}$ given by

$$f_n(x) = \frac{3x^2}{x^2 + \left(1 - n\,x\right)^2}$$

(a) Prove that there exists a function $f\colon [0,1] \to \mathbb{R}$ such that the sequence f_n converges pointwise to f on $[0,1]$. *Hint: First, find $f(x)$. Then prove $f_n \to f$ pointwise.*

(b) Does f_n converge to f uniformly? *Hint: Compute $f_n\left(\frac{1}{n}\right)$.*

(c) Is the sequence f_n uniformly bounded? *Hint: $a^2 \le a^2 + b^2$ for all $a, b \in \mathbb{R}$.*

(d) Is the family $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ equicontinuous? *Hint: Arzelà-Ascoli Theorem.*

Solution:

(a) Let $f\colon [0,1] \to \mathbb{R}$ be the constant function $f(x) = 0$. For any $0 < x \le 1$, we have:

$$\lim_{n\to+\infty} \frac{3x^2}{x^2 + \left(1 - n\,x\right)^2} = \lim_{n\to+\infty} \frac{3}{1 + \left(\frac{1}{x} - n\right)^2} = 0,$$

because $\left(\frac{1}{x} - n\right)^2 \to +\infty$ as $n \to +\infty$. Moreover, at $x = 0$, we have $f_n(0) = 0$ for all $n \in \mathbb{N}$. Thus, $f_n \to f$ pointwise for all $x \in [0,1]$.

(b) **No.** If $f_n \to f$ uniformly, then we would have that for all $\varepsilon > 0$, there would exist $N \in \mathbb{N}$ such that if $n \ge N$ then $|f_n(x)| < \varepsilon$ for all $x \in [0,1]$. However, this does not hold for any $\varepsilon < 3$, because $f_n\left(\frac{1}{n}\right) = 3$ for all $n \in \mathbb{N}$.

(c) **Yes.** For all $n \in \mathbb{N}$ and $0 < x \le 1$, we have $x^2 + \left(1 - n\,x\right)^2 \ge x^2 > 0$ and hence

$$|f_n(x)| = \left|\frac{3x^2}{x^2 + \left(1 - n\,x\right)^2}\right| \le 3.$$

Moreover, $|f_n(0)| = 0 \le 3$ as well. So f_n are uniformly bounded by $M = 3$.

(d) **No.** Since f_n is uniformly bounded by (c), if $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ was equicontinuous, then the Arzelà-Ascoli Theorem would imply that f_n has a uniformly convergent subsequence f_{n_k}. Such a subsequence f_{n_k} would also converge to the (pointwise) limit f of the sequence f_n. However, by the same argument in (b), we know that $f_{n_k}\left(\frac{1}{n_k}\right) = 3$ for all $k \in \mathbb{N}$, so this subsequence cannot converge uniformly to f.

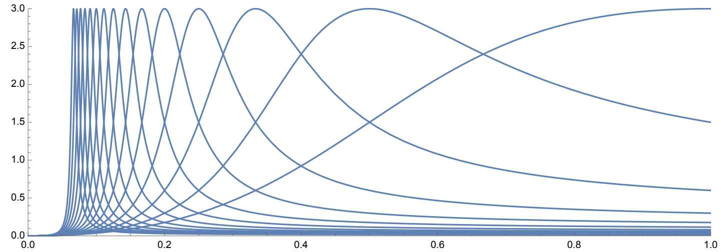

Figure 1: The graphs of $f_n(x)$ for $1 \le n \le 15$.