Today:

1. Introduction / Course logistics & syllabus
2. Natural numbers and proofs by induction (§1 of Ross).

Natural numbers: $\mathbb{N} = \{1, 2, 3, 4, \dots\}$

Axioms

1 is an element of the set $\mathbb{N}$

N1. $1 \in \mathbb{N}$

(N4)

N2. If $n \in \mathbb{N}$, then there exists a unique successor of n, denoted $n+1$, such that $n+1 \in \mathbb{N}$.

N3. The element $1 \in \mathbb{N}$ is not the successor of any element in $\mathbb{N}$.

N4. If $n, m \in \mathbb{N}$ such that their successors coincide, i.e., $n+1 = m+1$, then $n = m$.

N5. If a subset $S \subset \mathbb{N}$ is such that

- $1 \in S$
- If $n \in S$, then $n+1 \in S$

then $S = \mathbb{N}$.

Important for proving statements by induction.

Proofs by Induction ← based on Axiom N5.

Suppose you wish to prove that a certain property, say P_n, holds for all $n \in \mathbb{N}$.
Define the set $S = \{n \in \mathbb{N} : P_n \text{ holds}\}$.
By Axiom N5, to prove that $S = \mathbb{N}$, it suffices to show that:

- **Basis:** P_1 holds, i.e., $1 \in S$.
- **Step:** If P_n holds, then P_{n+1} also holds, i.e., if $n \in S$, then $n+1 \in S$.

Example:

Proposition: For all $n \in \mathbb{N}$,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Proof: (by Induction) Let P_n be the property that

$$P_n: \quad 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Basis: P_1: $1 = \frac{1.(1+1)}{2}$ ✓ true.

Step: We need to show that if P_n is true, Then also P_{n+1} is true.

Since we are assuming that P_n is true, we have

$$1+2+\dots+n = \frac{n(n+1)}{2}$$

Add $n+1$ to both sides:

$$1+2+\dots+n+(n+1) \overset{(P_n)}{=} \frac{n(n+1)}{2} + (n+1)$$
$$= \frac{(n+1)}{2}\left(n+2\right)$$
$$= \frac{(n+1)((n+1)+1)}{2}$$

The above statement is precisely P_{n+1}.
By induction, it follows that P_n is true for all $n \in \mathbb{N}$. $\square$

<u>Proposition</u>: For all $n \in \mathbb{N}$, the number $5^n - 4n - 1$ is divisible by 16.

<u>Pf.</u> (by Induction). Let P_n be the property that

$$\underbrace{16 \mid 5^n - 4n - 1.}_{\text{"}5^n-4n-1 \text{ is divisible by } 16\text{"}}$$

"$a \mid b$ means b is divisible by a"

Basis: P_1: $5^1 - 4 \cdot 1 - 1 = 0$ ✓

Step: $P_n \Longrightarrow P_{n+1}$

"implies"

Suppose $16 \mid 5^n - 4n - 1$. Now compute

$$5^{n+1} - 4(n+1) - 1 = 5 \cdot 5^n - 4n - 4 - 1$$

$$-16n \qquad +16n$$

$$= 5 \cdot 5^n - 20n - 5 + 16n$$

$$= 5\big(\underbrace{5^n - 4n - 1}\big) + \underbrace{16n}$$

by P_n, we know that this number is divisible by 16

this is also divis. by 16.

The above number is a sum of two numbers divisible by 16, thus it is divisible by 16 as well. This verifies that P_{n+1} holds, provided that P_n does. $\square$

<u>Proposition</u>: (Ex 1.3). For all $n \in \mathbb{N}$,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

<u>Proof</u>: Let P_n be the property that

$$P_n: \quad 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

<u>Basis</u>: P_1 ; $\quad 1^2 = \dfrac{1(1+1)(2.1+1)}{6} = \dfrac{2.3}{6}$ ✓

<u>Step</u>: Want to show that $P_n \Longrightarrow P_{n+1}$.

$$1^2 + 2^2 + \cdots + n^2 + (n+1)^2 \overset{(P_n)}{=} \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$= \frac{(n+1)}{6}\Big(\underbrace{n(2n+1) + 6(n+1)}_{\substack{2n^2 + n + 6n + 6 \\ = (n+2)(2n+3)}}\Big)$$

$$= \frac{(n+1)(n+2)(2n+3)}{6}$$

The above is precisely P_{n+1}. $\square$

<u>Exercise</u>: Try on your own!

Prove that for all $n \in \mathbb{N}$, $\quad 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\dfrac{n(n+1)}{2}\right)^2$

using induction.

Note: There are similar formulas for

$$1^k + 2^k + 3^k + \dots + n^k = \dots$$

in terms of Bernoulli numbers. These formulas are difficult to obtain, but, once you have a candidate, proving it by induction is easy.

Ex: Prove that for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$

$$|\sin(nx)| \leq n \cdot |\sin x|.$$

(this is done in Ross.)

Proposition: (Ex 1.6). For all $n \in \mathbb{N}$, $7 \mid (11^n - 4^n)$.

Proof: (by Induction)

Let P_n be the statement $7 \mid (11^n - 4^n)$.

Basis: P_1: $7 \mid 11^1 - 4^1$ ✓

Step: $P_n \Rightarrow P_{n+1}$.

$$\begin{aligned} 11^{n+1} - 4^{n+1} &= 11 \cdot 11^n - 4 \cdot 4^n \\ &= (4+7) \cdot 11^n - 4 \cdot 4^n \\ &= 4 \cdot (\underbrace{11^n - 4^n}) + \underbrace{7 \cdot 11^n} \end{aligned}$$

is divisible by 7 by P_n

is also clearly divisible by 7!

The above sum of numbers div. by 7 is also div. by 7. $\square$

Proposition: For all $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$,

$$b \mid (a+b)^n - a^n$$

Note: The previous Proposition was the case $a=4$, $b=7$.

These also follow from the

$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$

Binomial Theorem: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k \cdot b^{n-k}$

$$= \binom{n}{n} a^n + \binom{n}{1} a^{n-1} \cdot b + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{2} a^2 b^{n-2} + \binom{n}{1} a b^{n-1} + \binom{n}{n} b^n$$

This Thm can also be proved by induction on n.

Note:

$$(a+b)^n - a^n = \cancel{a^n} + \binom{n}{1} a^{n-1} \underline{b} + \cdots + \binom{n}{1} a \underline{b}^{n-1} + \underline{b}^n - \cancel{a^n}$$

$$= b\Bigg(\underbrace{\binom{n}{1} a^{n-1} + \binom{n}{2} a^{n-2} b + \cdots + \binom{n}{1} a b^{n-2} + b^{n-1}}_{\in \mathbb{Z} \text{ if } a, b \in \mathbb{Z}.}\Bigg)$$