Exercise session (Highlighted problems were solved)

1) Prove that $\liminf_{n\to\infty}(s_n+t_n) \geq \liminf_{n\to\infty} s_n + \liminf_{n\to\infty} t_n$

Give an example where the inequality is strict.

2) Justify whether each of the following series converges or diverges:

$$\sum_{n=1}^{+\infty} \frac{5^n}{n!}, \quad \sum_{n=0}^{+\infty}\left(\frac{2}{(-1)^n-3}\right)^n, \quad \frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^3}+\dots$$

3) TRUE or FALSE?

If TRUE, give a complete proof.
If FALSE, give a counter-example.

- If $(s_n)_{n\in\mathbb{N}}$ converges, then $(|s_n|)_{n\in\mathbb{N}}$ converges.
- If $(|s_n|)_{n\in\mathbb{N}}$ converges, then $(s_n)_{n\in\mathbb{N}}$ converges.
- If $s_n \to 0$ and $t_n \to +\infty$, then $s_n \cdot t_n \to 1$.
- If $\sum_{n=1}^{+\infty} a_n$ converges and $a_n \geq 0$, then $\sum_{n=1}^{+\infty} a_n^2$ converges
- If $\sum_{n=1}^{+\infty} a_n^2$ converges and $a_n \geq 0$, then $\sum_{n=1}^{+\infty} a_n$ converges

1) Prove that $\liminf_{n\to\infty}(s_n+t_n) \geq \liminf_{n\to\infty} s_n + \liminf_{n\to\infty} t_n$

Give an example where the inequality is strict.

Recall: $\liminf_{n\to\infty} a_n = \lim_{N\to\infty} \inf\{a_n : n>N\}$

Sol: $$\liminf_{n\to\infty}(s_n+t_n) = \lim_{N\to\infty} \underbrace{\inf\{s_n+t_n : n>N\}}_{c_N}$$

$$\liminf_{n\to\infty} s_n = \lim_{N\to\infty} \underbrace{\inf\{s_n : n>N\}}_{a_N}$$

$$\liminf_{n\to\infty} t_n = \lim_{N\to\infty} \underbrace{\inf\{t_n : n>N\}}_{b_N}$$

$$a_N = \inf\{s_n : n>N\}$$

$$b_N = \inf\{t_n : n>N\}$$

$$c_N = \inf\{s_n+t_n : n>N\}$$ ← largest lower bound for s_n+t_n, $n>N$

$$\begin{cases} a_N \leq s_n & \forall n>N \\ b_N \leq t_n & \forall n>N \end{cases} \Longrightarrow a_N+b_N \leq s_n+t_n \quad \forall n>N$$

↖ some lower bound for t_n+s_n, $n>N$

Thus $a_N + b_N \leq c_N$. Taking the limit as $N \to \infty$

$$\lim_{N\to\infty} (a_N + b_N) \leq \lim_{N\to\infty} c_N = \liminf_{n\to\infty} (s_n + t_n)$$

$$\| $$

$$\lim_{N\to\infty} a_N + \lim_{N\to\infty} b_N$$

$$\|$$

$$\liminf_{n\to\infty} s_n + \liminf_{n\to\infty} t_n .$$

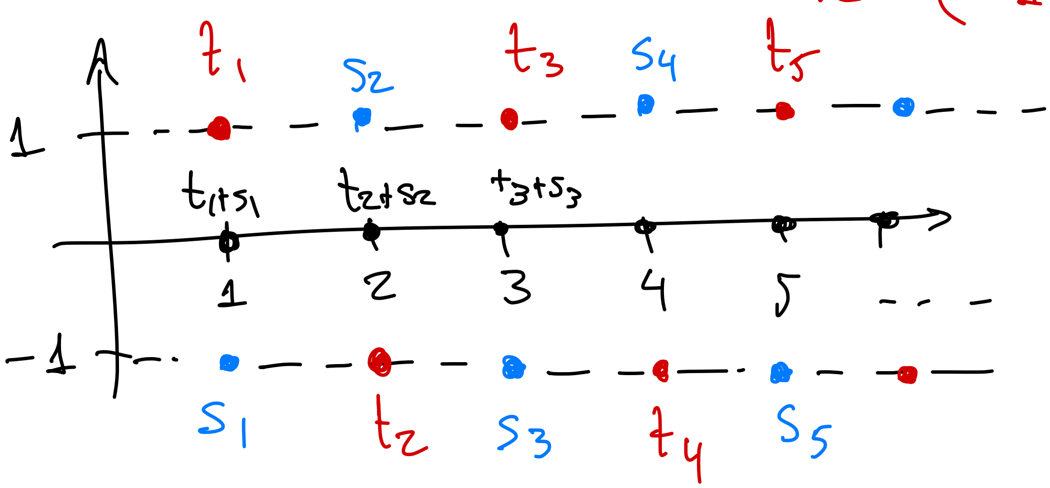
Limit of a sum of converging sequences is equal to the sum of their limits

$\square$

For example, consider $s_n = (-1)^n = \begin{cases} 1 & \text{if } n \text{ even} \\ -1 & \text{if } n \text{ odd} \end{cases}$

$s_n + t_n = 0$

$t_n = -s_n = (-1)^{n+1} = \begin{cases} 1 & \text{if } n \text{ odd} \\ -1 & \text{if } n \text{ even} \end{cases}$



$$\liminf_{n\to\infty} s_n = \liminf_{n\to\infty} t_n = -1$$

$$\limsup_{n\to\infty} s_n = \limsup_{n\to\infty} t_n = 1$$

$$\liminf_{n\to\infty}(t_n+s_n) = \lim_{n\to\infty} 0 = 0 > \underbrace{\liminf_{n\to\infty} t_n}_{-1} + \underbrace{\liminf_{n\to\infty} s_n}_{-1}$$

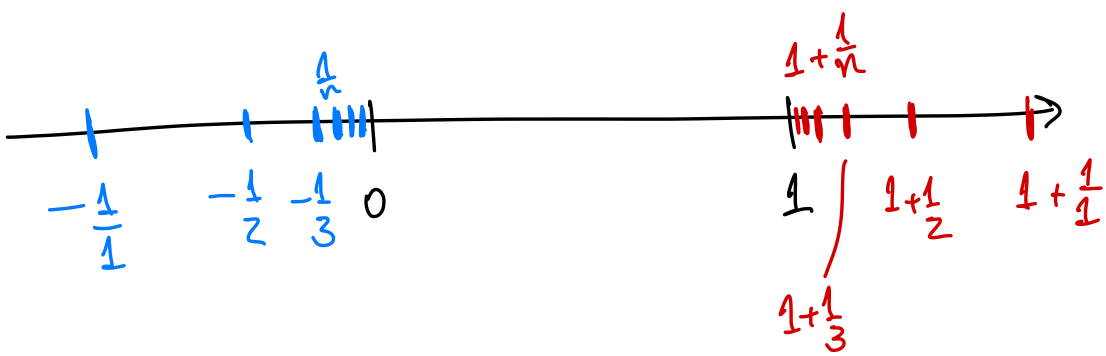
$$= -2.$$

HW2 Prob #1.

$$\lim_{n\to\infty} -\frac{1}{n} = 0$$

$$\lim_{n\to\infty} 1+\frac{1}{n} = 1$$

$$E = \bigcap_{n=1}^{\infty}\left[-\frac{1}{n},\ 1+\frac{1}{n}\right]$$



$$\underset{n=1}{[-1,2]} \supsetneq \underset{n=2}{\left[-\frac{1}{2},\frac{3}{2}\right]} \supsetneq \underset{n=3}{\left[-\frac{1}{3},\frac{4}{3}\right]} \supsetneq \cdots \supseteq [0,1]$$

$E = [0,1]$ is bounded, $\inf E = \min E = 0$,

$\sup E = \max E = 1.$

TRUE or FALSE?

If TRUE, give a complete proof.

If FALSE, give a counter-example.

$\boxed{\text{TRUE}}$ If $(s_n)_{n\in\mathbb{N}}$ converges, then $(|s_n|)_{n\in\mathbb{N}}$ converges.

$\boxed{\text{FALSE}}$ If $(|s_n|)_{n\in\mathbb{N}}$ converges, then $(s_n)_{n\in\mathbb{N}}$ converges.

$\boxed{\text{FALSE}}$ If $s_n \to 0$ and $t_n \to +\infty$, then $s_n \cdot t_n \to 1$.

hypothesis conclusion

Expectation: $s_n \to L \implies |s_n| \to |L|$.

$\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ $\qquad$ $\forall \varepsilon > 0\ \exists N' \in \mathbb{N}$

$n > N \implies |s_n - L| < \varepsilon$ $\qquad$ $n > N' \implies \big||s_n| - |L|\big| < \varepsilon$

Need to relate $|s_n - L|$ and $\big||s_n| - |L|\big|$ ...

What inequality relates $|a-b|$ and $\big||a|-|b|\big|$?

<u>Claim</u>: $\big||a| - |b|\big| \le |a-b|$.

<u>Pf</u>:

triangle ineq

$$|a| = |\underbrace{a-b}_{x} + \underbrace{b}_{y}| \le |\underbrace{a-b}_{x}| + |\underbrace{b}_{y}| \implies |a| - |b| \le |a-b|$$

$$|b| = |\underbrace{b-a}_{x} + \underbrace{a}_{y}| \le |a-b| + |a| \implies -\big(|a| - |b|\big) \le |a-b|$$

As $|z| = \max\{z, -z\}$, it follows that

$$\big||a| - |b|\big| \leq |a-b|. \qquad \square$$

Given $\varepsilon > 0$, let $N \in \mathbb{N}$ be such that $n > N$ implies $|s_n - L| < \varepsilon$. Then, by the Claim above, with $a = s_n$, $b = L$, we have

$$\big||s_n| - |L|\big| \leq |s_n - L| < \varepsilon;$$

i.e., $|s_n|$ converges to $|L|$. $\square$

2.

Counter-example to:

"If $(|s_n|)_{n\in\mathbb{N}}$ converges, then $(s_n)_{n\in\mathbb{N}}$ converges."

$|s_n| \longrightarrow 1$, but $(s_n)_{n\in\mathbb{N}}$ does not converge.

Consider $s_n = (-1)^n$. Note that $|s_n| = |(-1)^n| = 1$; so $|s_n| \to 1$. However $(s_n)_{n\in\mathbb{N}}$ does not converge; since $-1 = \liminf_{n\to\infty} s_n \neq \limsup_{n\to\infty} s_n = +1$.

(FALSE) If $s_n \to 0$ and $t_n \to +\infty$, then $s_n \cdot t_n \to 1$.

Counter-example:

Take $s_n = \frac{2}{n}$ and $t_n = n$. Clearly, $s_n \to 0$ and $t_n \to +\infty$; however, $s_n \cdot t_n = \frac{2}{n} \cdot n = 2$, so $s_n \cdot t_n \longrightarrow 2 \neq 1$.

Follow-up: Can we have $s_n \to 0$, $t_n \to \infty$, and $s_n \cdot t_n \to +\infty$?

Yes: $s_n = \frac{1}{n}$, $t_n = n^2$.

$s_n \to 0$, $t_n \longrightarrow +\infty$

and $s_n \cdot t_n = \frac{1}{n} \cdot n^2 = n \longrightarrow +\infty$.