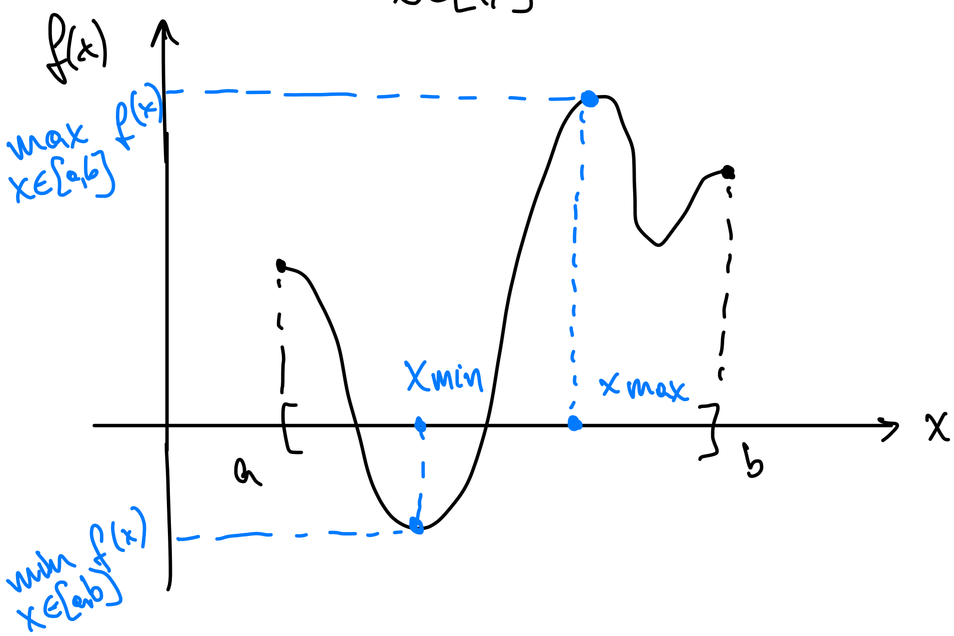
Recall: $f: D \subset \mathbb{R} \to \mathbb{R}$ is continuous at x_0 if for every sequence $x_n \to x_0$, the sequence $f(x_n)$ satisfies $\lim_{n\to\infty} f(x_n) = f(x_0)$.

Very important in optimization, to guarentee the existence of min/max.

Theorem. Let $f: [a,b] \to \mathbb{R}$ be a continuous function. Then $f(x)$ is bounded, and $f(x)$ assumes its min and max in $[a,b]$, i.e., there exist points $x_{min}, x_{max} \in [a,b]$ such that

$$f(x_{min}) = \min_{x\in[a,b]} f(x) \quad \text{and} \quad f(x_{max}) = \max_{x\in[a,b]} f(x).$$



Proof: Suppose that $f: [a,b] \to \mathbb{R}$ is unbounded, that is, $\forall n \in \mathbb{N}$ $\exists x_n \in [a,b]$ s.t. $|f(x_n)| \geq n$.

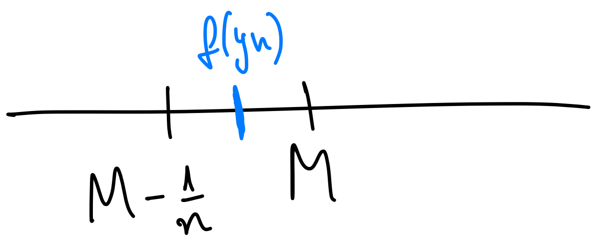
By the Bolzano–Weierstrass Theorem, the sequence x_n must admit a convergent subsequence x_{n_k} because it is bounded; say $x_{n_k} \longrightarrow x_0 \in [a,b]$.

Since $f(x)$ is continuous, $f(x_{n_k}) \longrightarrow f(x_0)$.

However $|f(x_{n_k})| \geq n_k$, so $|f(x_{n_k})| \longrightarrow +\infty$, contradicting $|f(x_{n_k})| \longrightarrow |f(x_0)| < +\infty$.

Thus, $f(x)$ is bounded on $[a,b]$.

Let $M = \sup \{ f(x) : x \in [a,b] \}$. Since $f(x)$ is bounded, this sup exists and is a finite real number. For every $n \in \mathbb{N}$, there exists $y_n \in [a,b]$ such that



$$M - \frac{1}{n} < f(y_n) \leq M.$$

Clearly, $\lim_{n \to \infty} f(y_n) = M$.

By Bolzano-Weierstrass, y_n has a convergent subsequence y_{n_k}, say $y_{n_k} \longrightarrow x_{max}$. Since $f(x)$ is continuous

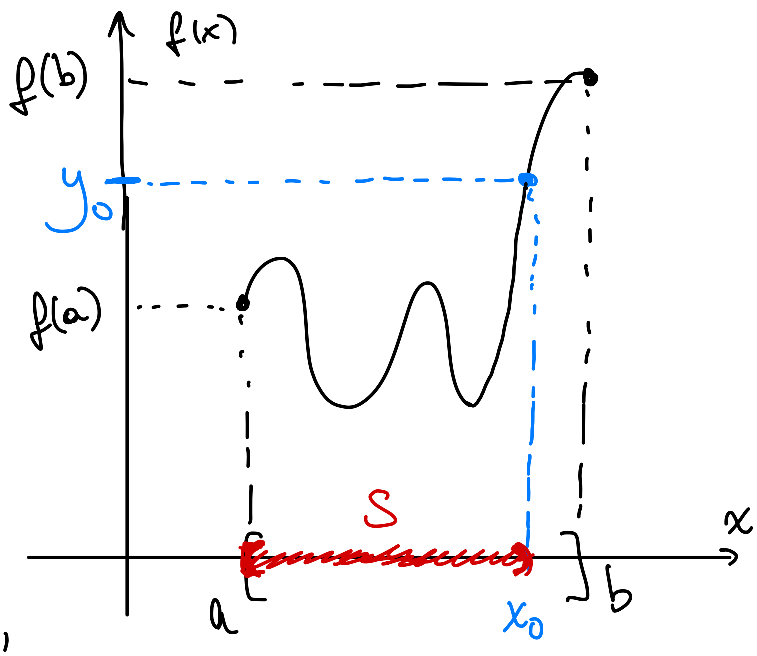
$y_{n_k} \longrightarrow x_{max} \implies f(y_{n_k}) \to f(x_{max})$ and therefore $f(x_{max}) = M$. In other words, $\sup_{x \in [a,b]} f(x)$ is attained at $x = x_{max}$. Similarly, applying the same reasoning to $-f(x)$ we find $x_{min} \in [a,b]$ s.t.

$-f(x_{min}) = \sup_{x\in[a,b]} -f(x)$ and thus $f(x_{min}) = \inf_{x\in[a,b]} f(x)$,

i.e., $\inf_{x\in[a,b]} f(x)$ is attained at $x = x_{min}$. $\square$

Intermediate Value Theorem.

If $f:[a,b] \to \mathbb{R}$ is a continuous function, and y_0 is between $f(a)$ and $f(b)$, then there exists $x_0 \in [a,b]$ such that $f(x_0) = y_0$.



Proof: Without loss of generality, let us assume that $f(a) < y_0 < f(b)$. Let

$$S = \{x \in [a,b] : f(x) < y_0\}.$$

Since $a \in S$, we have $S \neq \emptyset$; let $x_0 = \sup S$.

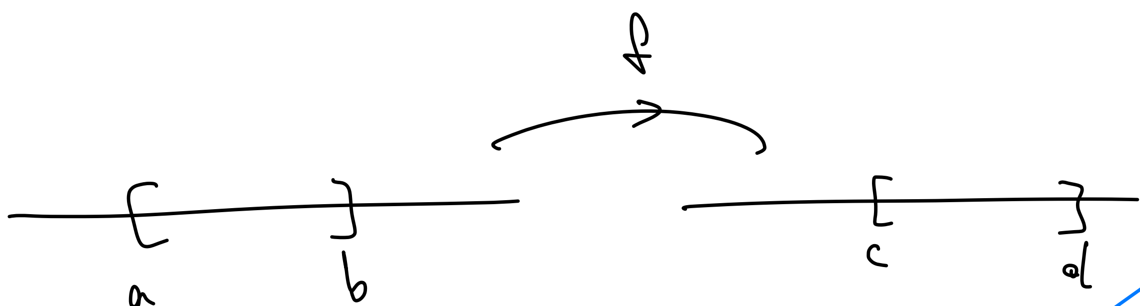
Thus $\forall n \in \mathbb{N}$, $\exists\, s_n \in S$ s.t. $x_0 - \frac{1}{n} < s_n \leq x_0$.

Clearly $s_n \to x_0$ and $f(s_n) < y_0$ for all $n \in \mathbb{N}$;

so $f(x_0) = \lim_{n\to\infty} f(s_n) \leq y_0$. Let $t_n = \min\{b, x_0 + \frac{1}{n}\}$, Then $x_0 \le t_n \le x_0 + \frac{1}{n}$ so $t_n \to x_0$ and $t_n \notin S$, $\forall n \in \mathbb{N}$ so $f(t_n) \geq y_0$, $\forall n \in \mathbb{N}$. Thus $f(t_n) \to f(x_0)$ and $f(x_0) = \lim_{n\to\infty} f(t_n) \geq y_0$.

Therefore $y_0 \le f(x_0) \le y_0$ so $f(x_0) = y_0$. $\square$

<u>Corollary</u>. The image $\{f(x) : x \in [a,b]\}$ of a closed interval $[a,b]$ by a continuous function $f : [a,b] \to \mathbb{R}$ is also a closed interval, or a single point.

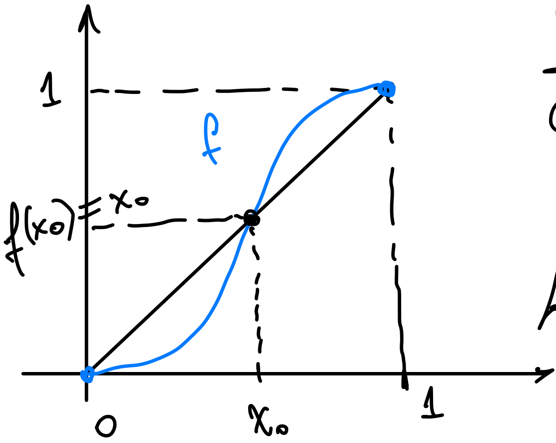


From the first theorem proven today!

(More precisely, $\{f(x) : x \in [a,b]\} = [f(x_{min}), f(x_{max})]$

<u>Applications of Intermediate Value Theorem</u>

1. Existence of <u>fixed points</u>: If $f : [0,1] \to [0,1]$ is a continuous function mapping $[0,1]$ to itself, then there exists $x_0 \in [0,1]$ a <u>fixed point</u>, i.e., $f(x_0) = x_0$.



Pf: Let $g(x) = f(x) - x$; note $g(x)$ is continuous,

$$g(0) = f(0) \geq 0$$
$$g(1) = f(1) - 1 \leq 0.$$

Apply I.V.T. to $g(x)$ and $y_0 = 0$, we see that $\exists x_0 \in [0,1]$ s.t. $g(x_0) = y_0 = 0$.
Thus, $f(x_0) = x_0$. $\square$

2. Existence of n^{th} root $\sqrt[n]{a}$ of any $a>0$

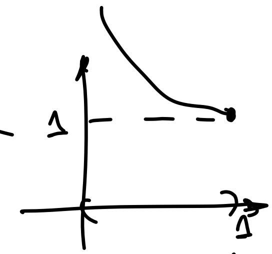
Consider $f(x) = x^n - a$, which is a continuous function and $f(x_0) = 0$ precisely at $x_0 = \sqrt[n]{a}$. To prove that such x_0 exists, we use I.V.T. with:

$$f(0) = 0^n - a = -a < 0$$
$$f(b) = b^n - a > 0$$

for every $b > 0$ such that $a < b^n$. Therefore I.V.T. implies $\exists x_0 \in (0,b)$ such that $f(x_0) = 0$, as desired.

Exercise: TRUE or FALSE? Justify.

1. A continuous function attains a maximum in any interval.



not closed!

FALSE: consider $f:(0,1)\to\mathbb{R}$, given by $f(x)=\frac{1}{x}$.
$f(x)$ is continuous at every $x_0\in(0,1)$, but
$\lim_{x\to 0} f(x) = +\infty$; there's no maximum.

(To make it TRUE, we must request that the interval be closed)

2. The image of any function $f:[a,b]\to\mathbb{R}$ is an interval.

FALSE: Take for example

$$f(x)=\begin{cases} 0 & \text{if } x\in\left[a,\frac{a+b}{2}\right) \\ 1 & \text{if } x\in\left[\frac{a+b}{2},b\right]\end{cases}$$

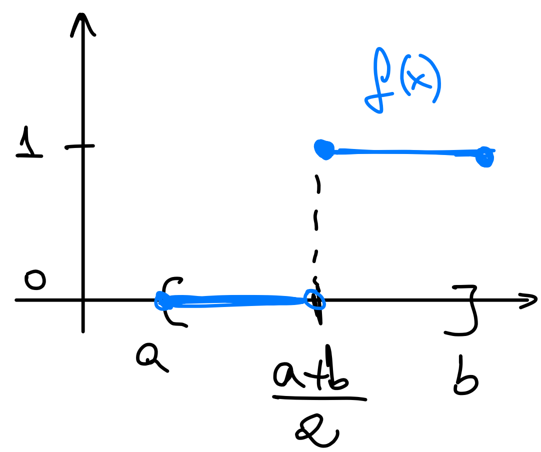


Image of $f(x)$ is $\{0,1\}$, so not an interval.

(To make it TRUE, we must request $f(x)$ to be continuous)

3. Every polynomial of odd degree has at least one real root. What happens if the degree is even?

TRUE: Application of Intermediate Value Theorem.

(odd degree) Proof: Let $p(x)=a_nx^n+a_{n-1}x^{n-1}+\dots+a_1x+a_0$
(Up to replacing p with $-p$, without loss of generality, assume $a_n>0$)

Since $n = \text{degree } p(x)$ is odd, we have that for x sufficiently large negative, $p(x) < 0$

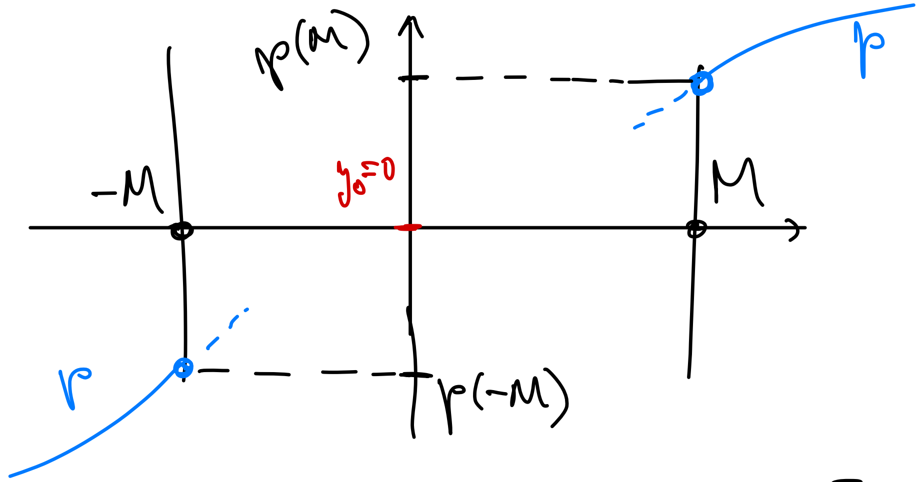
x sufficiently large positive, $p(x) > 0$,

i.e. $\lim_{x \to -\infty} p(x) = -\infty$ and $\lim_{x \to +\infty} p(x) = +\infty$.

More details on how to justify this when we study limits of functions.

Thus $\exists M > 0$ such that

$$p(-M) < 0 \quad \text{and} \quad p(M) > 0$$

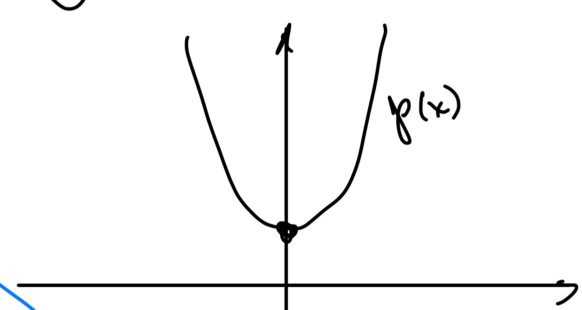


By the Intermediate Value Theorem, $\exists x_0 \in [-M, M]$ such that $p(x_0) = 0$.

Q: What about polynomials of even degree?

A: They may not have any real roots:

$$p(x) = x^2 + 1.$$



(What fails is that $\lim_{x \to -\infty} p(x) = +\infty$ if n is even.)