More uniform convergence & Weierstrass M-test.

Recall: $f_n \to f$ uniformly on S if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t.

$$n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon \quad \text{for all } x \in S.$$

"Graphs of f_n converge to the graph of f over S"

f_n cont. on $S \Rightarrow f$ is cont. on S

f_n diff. on $S \not\Rightarrow f$ is diff. on S.

Thm. If $f_n \to f$ uniformly on $S = [a,b]$ and f_n is continuous for all $n \in \mathbb{N}$, then:

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Pf: Since f_n and f are continuous, so are $f_n - f$ (for all $n \in \mathbb{N}$.) In particular, $f_n - f$ are integrable.

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| = \left| \int_a^b (f_n(x) - f(x))\,dx \right|$$

Triangle inequality (for integrals)

$$\leq \int_a^b |f_n(x) - f(x)|\,dx$$

Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ s.t. $n \geq N$

$$|f_n(x) - f(x)| < \frac{\varepsilon}{b-a} \quad \text{for all } x \in [a,b]$$

Thus, for $n \geq N$, we have

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| \leq \int_a^b \underbrace{|f_n(x) - f(x)|}_{< \varepsilon/(b-a)}\,dx$$

$$< \int_a^b \frac{\varepsilon}{b-a}\,dx = \frac{\varepsilon}{b-a} \underbrace{\int_a^b dx}_{= x\big|_a^b = b-a} = \frac{\varepsilon}{b-a} \cdot (b-a) = \varepsilon$$

We conclude that if $n \geq N$, then $\left| \underbrace{\int_a^b f_n(x)\,dx}_{S_n} - \underbrace{\int_a^b f(x)\,dx}_{L} \right| < \varepsilon$,

i.e. $\int_a^b f_n(x)\,dx \longrightarrow \int_a^b f(x)\,dx$. $\square$

<u>Example</u>: Compute the following limit:

$$\lim_{n\to\infty} \underbrace{\int_0^1 \frac{nx + \sin(nx^2)}{n}\,dx}_{\text{cannot be computed explicitly as a sequence } s_n \text{ because } \int \sin(x^2)dx \text{ cannot be expressed in elementary terms.}} = \lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx$$

b/c $f_n \to f$ uniformly on $[0,1]$

$$f_n(x) = \frac{nx + \sin(nx^2)}{n}$$ converges uniformly to $f(x) = x$;

because: $f_n(x) = x + \frac{\sin(nx^2)}{n} = f(x) + \frac{\sin(nx^2)}{n}$

$$\forall x, \quad |f_n(x) - f(x)| = \left|\frac{\sin(nx^2)}{n}\right| \leq \frac{1}{n} < \varepsilon$$

if $n > \lceil 1/\varepsilon \rceil =: N$; i.e. $(f_n - f) \to 0$ uniformly

hence $f_n \to f$ uniformly.

By the above result, we find:

$$\lim_{n\to\infty} \underbrace{\int_0^1 \frac{nx + \sin(nx^2)}{n}\, dx}_{\int_0^1 f_n(x)\,dx} = \underbrace{\int_0^1 x\, dx}_{\int_0^1 f(x)\,dx} = \boxed{\frac{1}{2}}.$$

Q: How can we ensure f_n converges uniformly to some f?

Definition: A sequence (f_n) is uniformly Cauchy on S if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $n, m \geq N \Rightarrow |f_n(x) - f_m(x)| < \varepsilon$ for all $x \in S$.

Theorem. If (f_n) is uniformly Cauchy on S, then there exists a function f such that $f_n \to f$ uniformly.

Pf: (1) "Find f". For each $x_0 \in S$, we have that

$(f_n(x_0))_{n\in\mathbb{N}}$ is a Cauchy sequence. Since Cauchy seq. are convergent, we have that $\exists y_0 = \lim_{n\to\infty} f_n(x_0)$.
Define $f(x_0) = y_0$; for all choices of $x_0 \in S$. This defines a function f such that $f_n \to f$ pointwise.

(2) Prove $f_n \to f$ uniformly. Since (f_n) is uniformly Cauchy, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $n, m \geq N \Rightarrow |f_n(x) - f_m(x)| < \frac{\varepsilon}{2}$
Taking the limit as $m \nearrow +\infty$ in $|f_n(x) - f_m(x)| < \frac{\varepsilon}{2}$ we see that $|f_n(x) - f(x)| \leq \frac{\varepsilon}{2} < \varepsilon$. So, given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon$, that is $f_n \to f$ uniformly.

$\square$

<u>Thm</u> (<u>Weierstrass M-test</u>). Let $(M_k)_{k\in\mathbb{N}}$ be a sequence such that $\sum_{k=1}^{\infty} M_k < \infty$. If $g_k(x)$ is a sequence of functions such that $|g_k(x)| \leq M_k$ for all $x \in S$ and $k \in \mathbb{N}$, then $\sum_{k=1}^{\infty} g_k(x)$ converges uniformly on S.

i.e., the sequence of partial sums $f_n(x) = \sum_{k=1}^{n} g_k(x)$ converges uniformly on S to $f(x) = \sum_{k=1}^{\infty} g_k(x)$.

<u>Cor</u>: If $g_k(x)$ are as above and continuous on S, then $f(x) = \sum_{k=1}^{\infty} g_k(x)$ is continuous on S.

<u>Pf</u> (Weierstrass M-test). Consider the sequence $f_n(x) = \sum_{k=1}^{n} g_k(x)$. By the results above, it suffices to prove that (f_n) is uniformly Cauchy. Since $\sum_{k=1}^{\infty} M_k < \infty$, the sequence $s_n = \sum_{k=1}^{n} M_k$ of partial sums is Cauchy. So, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ s.t. if $n > m \geq N$ then $|s_n - s_m| < \varepsilon$, i.e. $\sum_{k=m+1}^{n} M_k < \varepsilon$ because:

$$s_n - s_m = \sum_{k=1}^{n} M_k - \sum_{k=1}^{m} M_k$$

$$= (M_1 + \cdots + M_m + M_{m+1} + \cdots + M_n) - (M_1 + \cdots + M_m)$$

$$= \sum_{k=m+1}^{n} M_k$$

Since $|g_k(x)| \leq M_k$, it follows that, e.g.,

$$|f_n(x)| = \left|\sum_{k=1}^{n} g_k(x)\right| \underset{\substack{\uparrow \\ \text{triangle} \\ \text{ineq.}}}{\leq} \sum_{k=1}^{n} |g_k(x)| \leq \sum_{k=1}^{n} M_k$$

Similarly, if $n > m \geq N$,

$$|f_n(x) - f_m(x)| = \left| \sum_{k=1}^{n} g_k(x) - \sum_{k=1}^{m} g_k(x) \right|$$

$\sum_{k=1}^{n} M_k$ is Cauchy

$$= \left| \sum_{k=m+1}^{n} g_k(x) \right| \leq \sum_{k=m+1}^{n} |g_k(x)| \leq \sum_{k=m+1}^{n} M_k < \varepsilon$$

triangle ineq.

$|g_k(x)| \leq M_k$

so $f_n(x) = \sum_{k=1}^{n} g_k(x)$ is uniformly Cauchy, as desired. □

<u>Example</u>: Show that $f(x) = \sum_{k=1}^{+\infty} \frac{1}{x^2 + k^2}$ is a continuous function on $\mathbb{R}$.

Use Weierstrass M-test with $g_k(x) = \frac{1}{x^2 + k^2}$.

Clearly $|g_k(x)| = \left| \frac{1}{x^2 + k^2} \right| \leq \frac{1}{k^2}$. Set $M_k = \frac{1}{k^2}$.

$$\sum_{k=1}^{+\infty} M_k = \sum_{k=1}^{+\infty} \frac{1}{k^2} < \infty \quad (p\text{-Series } p=2)$$

Therefore $f_n(x) = \sum_{k=1}^{n} g_k(x)$ converges uniformly to $f(x) = \sum_{k=1}^{\infty} g_k(x)$.

In particular, $f(x)$ is the uniform limit of continuous functions hence continuous.