Recall from last time:

Thm (Weierstrass M-test). Let $(M_k)_{k\in\mathbb{N}}$ be a sequence such that $\sum_{k=1}^{\infty} M_k < \infty$. If $g_k(x)$ is a sequence of functions such that $|g_k(x)| \le M_k$ for all $x \in S$ and $k \in \mathbb{N}$, then $\sum_{k=1}^{\infty} g_k(x)$ converges uniformly on S.

Let $g(x) = \sum_{k=1}^{+\infty} g_k(x)$, where $g_k(x)$ are as above.

From earlier results (Continuity is preserved under uniform convergence), it follows that if $g_k(x)$ is continuous for all $k \in \mathbb{N}$, then so is $g(x)$.

Q: Domain of $g(x)$ if it is a power series; i.e., $g_k(x) = a_k x^k$ for all $k \in \mathbb{N}$?

A: If the radius of convergence is $R = \frac{1}{\beta}$, where $\beta = \limsup_{n\to\infty} |a_n|^{1/n}$, then Domain$(g)$ contains $(-R, R)$.

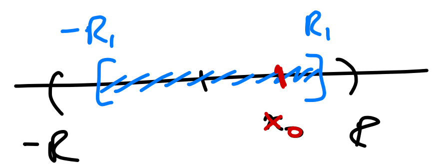
Proposition: If $\sum_{k=1}^{\infty} a_k x^k$ is a power series w/ radius of convergence R, then if $0 < R_1 < R$, the power series converges uniformly on $S = [-R_1, R_1]$ to a continuous function.

Pf: Let $0 < R_1 < R$, then note the radius of convergence of $\sum_{k=1}^{\infty} a_k x^k$ is the same as that of $\sum_{k=1}^{+\infty} |a_k| x^k$. Thus, as $R_1 < R$, we have $\sum_{k=1}^{+\infty} |a_k| R_1^k < \infty$

and $$\sum_{k=1}^{+\infty} |\underbrace{a_k x^k}_{g_k(x)}| = \sum_{k=1}^{+\infty} |a_k| \cdot |x|^k \leq \sum_{k=1}^{+\infty} \underbrace{|a_k| \cdot R_1^k}_{M_k} < \infty,$$

if $|x| \leq R_1$. Since $\sum_{k=1}^{\infty} M_k < \infty$, the power series $\sum_{k=1}^{+\infty} a_k x^k$ converges uniformly on $S = [-R_1, R_1]$ by the Weierstrass M-test. $\square$

Corollary: The function $g(x) = \sum_{k=1}^{+\infty} a_k x^k$ as above is continuous at any $x_0 \in (-R, R)$.



Pf: For any $x_0 \in (-R, R)$, there exists $0 < R_1 < R$ such that $x_0 \in [-R_1, R_1]$; so $g(x)$ is cont. at $x = x_0$ by the above. $\square$

Lemma: If $\sum_{n=0}^{+\infty} a_n x^n$ has radius of convergence R, then $\sum_{n=1}^{+\infty} n \cdot a_n \cdot x^{n-1}$ and $\sum_{n=0}^{+\infty} \frac{a_n}{n+1} x^{n+1}$ also have radius of convergence equal to R.

Pf: First, note that the radius of convergence of $\sum_{n=1}^{+\infty} n \cdot a_n x^{n-1}$ is the same as that of

$$x \cdot \left(\sum_{n=1}^{+\infty} n\, a_n x^{n-1} \right) = \sum_{n=1}^{+\infty} n \cdot a_n x^n.$$ The same holds for $\sum_{n=0}^{+\infty} \frac{a_n}{n+1} x^n$. So the radius of convergence for these series is $R = \frac{1}{\beta}$ where

$$\beta_{\text{derivative}} = \limsup_{n\to\infty} |n \cdot a_n|^{1/n} = \underbrace{\lim_{n\to\infty} n^{1/n}}_{=1} \cdot \underbrace{\limsup_{n\to\infty} |a_n|^{1/n}}_{\beta_{\text{original}}} = \beta_{\text{original}}$$

$$\beta_{\text{integral}} = \limsup_{n\to\infty} \left|\frac{a_n}{n+1}\right|^{1/n} = \underbrace{\lim_{n\to\infty} \frac{1}{(n+1)^{1/n}}}_{=1} \cdot \underbrace{\limsup_{n\to\infty} |a_n|^{1/n}}_{\beta_{\text{original}}} = \beta_{\text{original}}$$

$\square$

"Integration term-by-term":

Theorem. Suppose $f(x) = \sum_{n=0}^{+\infty} a_n x^n$ has radius of convergence R.
Then $$\int_0^x f(t)\,dt = \sum_{n=0}^{+\infty} \frac{a_n}{n+1} x^{n+1} \quad \text{for} \quad |x| < R.$$

Pf: As seen above, $f(x) = \sum_{n=0}^{+\infty} a_n x^n$ converges uniformly on $[-R_1, R_1]$ for any $0 < R_1 < R$. Thus for all $0 < x < R_1$:

if instead $-R_1 < x < 0$, then pf is analogous (left as exercise)

$$\lim_{n\to\infty} \int_0^x \underbrace{\left(\sum_{k=0}^{n} a_k t^k \right)}_{f_n(t)} dt \overset{\text{Lecture 17}}{=} \int_0^x f(t)\, dt$$

finite sum

$$\int_0^x f_n(t)\,dt = \int_0^x \left(\sum_{k=0}^{n} a_k t^k \right) dt = \sum_{k=0}^{n} \left(\int_0^x a_k t^k \right) dt =$$

$$= \sum_{k=0}^{n} a_k \left. \frac{t^{k+1}}{k+1} \right|_0^x = \sum_{k=0}^{n} a_k \left(\frac{x^{k+1}}{k+1} - \cancel{\frac{0^{k+1}}{k+1}}^{\,0} \right)$$

$$\boxed{= \sum_{k=0}^{n} \frac{a_k}{k+1} x^{k+1}}$$

Partial sum until n of the series $\sum_{k=0}^{+\infty} \frac{a_k}{k+1} x^{k+1}$

Taking limits as $n \to \infty$ in the above proves the desired result. $\square$

"Differentiation term-by-term"

Theorem. Suppose $f(x) = \sum_{n=0}^{+\infty} a_n x^n$ has radius of convergence R. Then $f'(x) = \sum_{n=1}^{+\infty} n \cdot a_n \cdot x^{n-1}$ for $|x| < R$.

⚠ Proof must not use $\lim_{n\to\infty} f_n'(x) \cancel{=} f'(x)$ if $f_n \to f$.

$\boxed{\text{FALSE}}$

(Recall that differentiability need not be preserved under uniform convergence, like, say, continuity.)

Pf: We will use integration term-by-term.

Let $g(x) = \sum_{n=1}^{+\infty} n \cdot a_n \cdot x^{n-1}$ and recall that the radius of convergence of $g(x)$ is the same as that of $f(x) = \sum_{n=0}^{+\infty} a_n \cdot x^n$. By Integration term-by-term, we have

$$\int_0^x g(t)\,dt = \sum_{n=1}^{+\infty} \left(\int n a_n t^{n-1} \right) dt = \sum_{n=1}^{+\infty} a_n t^n \Big|_0^x$$

$$= \sum_{n=1}^{+\infty} a_n x^n = f(x) - a_0.$$

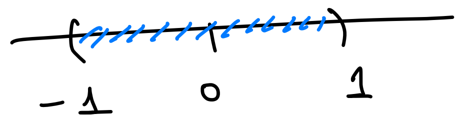
Therefore, as $f(x) = \int_0^x g(t)\,dt + a_0$, by the Fundamental theorem of Calculus $\frac{d}{dx}\int_0^x g(t)\,dt = g(x)$

so $f'(x) = g(x)$, for all $|x| < R$. $\square$

Revisiting an example from Lecture 15:

Heuristically / Informal

$$f(x) = \sum_{n=1}^{+\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$



Radius of convergence?

$$\beta = \limsup_{n\to\infty} \left|\frac{1}{n}\right|^{1/n} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = 1. \Longrightarrow R = \frac{1}{\beta} = 1.$$

From the results discussed above:

$f : (-1,1) \to \mathbb{R}$ is continuous at all $x_0 \in (-1,1)$.

and differentiable at all $x_0 \in (-1,1)$.

Differentiation term-by-term gives:

$$f'(x) = \sum_{n=1}^{+\infty} \frac{\cancel{n} x^{n-1}}{\cancel{n}} = \sum_{n=0}^{+\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$\Downarrow \qquad \underset{\text{Geometric series}}{=} \frac{1}{1-x} \quad \text{if } |x| < 1.$$

$$f(x) = \int \frac{dx}{1-x} = \ln\left(\frac{1}{1-x}\right).$$

Rmk: Integration term-by-term gives

$$\int f(x)\,dx = \sum_{n=1}^{+\infty} \frac{x^{n+1}}{n(n+1)} = \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} + \cdots$$