Limits

Def 1: If $f: S \subset \mathbb{R} \to \mathbb{R}$ and $x_0 \in S$, then

$\lim_{x \to x_0} f(x) = L$ if $\forall \varepsilon > 0$ $\exists \delta > 0$ such that

if $|x - x_0| < \delta$, then $|f(x) - L| < \varepsilon$.

Note: We do not need to assume $f(x)$ is defined at $x = x_0$

Def 2: If $f: S \subset \mathbb{R} \to \mathbb{R}$ and $x_0 \in S$, then

$\lim_{x \to x_0} f(x) = L$ if for all sequences (x_n) in S

such that $x_n \to x_0$, we have $f(x_n) \to L$.

Prop: Both definitions above are equivalent.

(1) $\Rightarrow$ (2) Let (x_n) be a sequence in S with $x_n \to x_0$.

Let $\varepsilon > 0$ be given. From (1) there exists $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - L| < \varepsilon$. Since $x_n \to x_0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|x_n - x_0| < \delta$, hence $|f(x_n) - L| < \varepsilon$.

(2) $\Rightarrow$ (1) Suppose that (2) holds but (1) fails:

$\exists \varepsilon > 0$ s.t. $\forall \delta > 0$, we have $|x - x_0| < \delta$ and $|f(x) - L| \geq \varepsilon$.

Choosing $\delta = \frac{1}{n}$ for each $n \in \mathbb{N}$, we have that $\exists\, x_n$ s.t. $|x_n - x_0| < \delta = \frac{1}{n}$ and $|f(x_n) - L| \geq \varepsilon$. Clearly $x_n \to x_0$ but $f(x_n) \not\to L$ since $f(x_n)$ is bounded away from L by $\varepsilon > 0$. This contradicts Def 2, so it implies that the assumption that (1) fails is wrong; so (1) holds as we wish to prove. $\square$

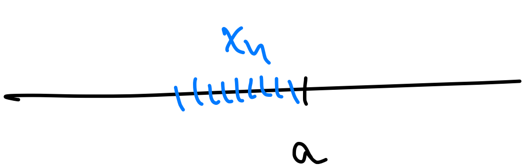
Lateral limits

$$\lim_{x \to a_-} f(x) = L$$

$a < x + \delta$

(1) $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. if $|x - a| < \delta$ and $x < a$ then $|f(x) - L| < \varepsilon$

(2) For all sequences $x_n \to a$ w/ $x_n < a$ for all $n \in \mathbb{N}$, $f(x_n) \to L$.

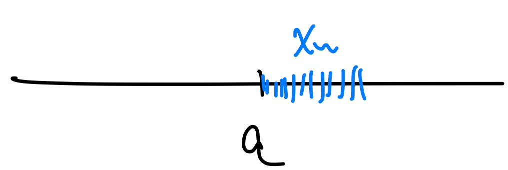


$$\lim_{x \to a_+} f(x) = M$$

$x < a + \delta$

(1) $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. if $|x - a| < \delta$ and $x > a$, then $|f(x) - M| < \varepsilon$.

(2) For all sequences $x_n \to a$ w/ $x_n > a$ for all $n \in \mathbb{N}$, $f(x_n) \to M$



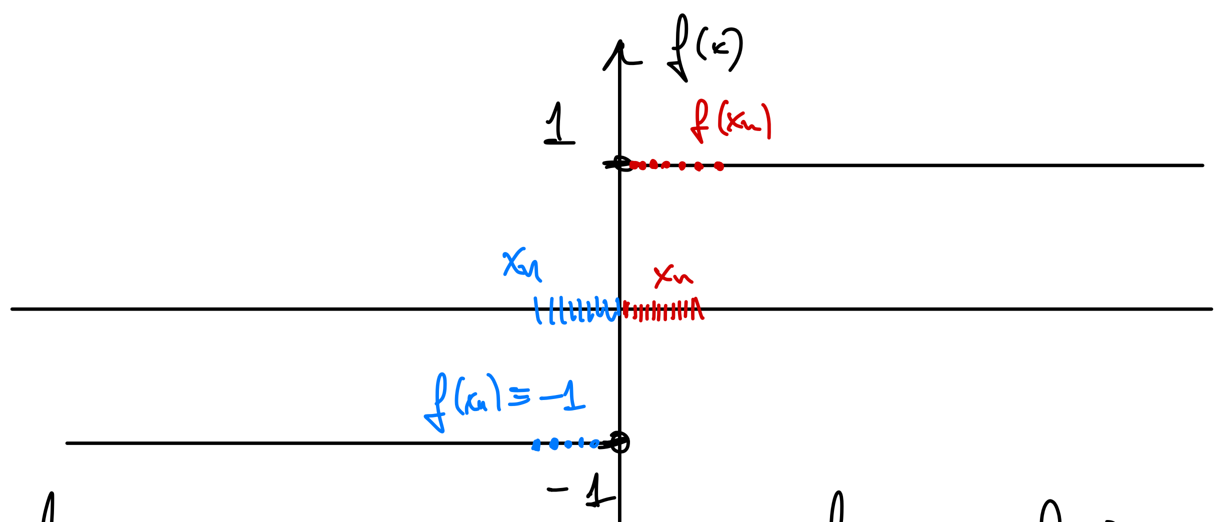
Prop: $\lim_{x\to a} f(x) = L$ if and only if both lateral limits exist and are equal to L, that is,

$$\lim_{x\to a_-} f(x) = L = \lim_{x\to a_+} f(x)$$

Pf: Exercise. (Using Def 1 or 2, as given above)

Ex: The above can be used to show that some limits do not exist: e.g., $f(x) = \frac{x}{|x|}$ for $x \neq 0$.

$$f(x) = \frac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0 \leftarrow |x| = x \\ -1 & \text{if } x < 0 \leftarrow |x| = -x \end{cases}$$



$$\lim_{x\to 0_-} f(x) = -1 \neq \lim_{x\to 0_+} f(x) = 1$$

So $\lim_{x\to 0} f(x)$ does not exist.

Derivatives

Def: A function $f : D \subset \mathbb{R} \to \mathbb{R}$ is differentiable at $x_0 \in D$ if the following limit exists:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

In case it exists, we write $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$

Example: $f(x) = x^2$, let $x_0 \in \mathbb{R}$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x \to x_0} (x + x_0) = 2x_0$$

$= (x - x_0) \cdot (x + x_0)$

So $f(x)$ is differentiable (at every $x_0 \in \mathbb{R}$), and $f'(x) = 2x$.

$x^n - x_0^n = (x - x_0)(x^{n-1} + \dots)$

Similarly: $f(x) = x^n \Longrightarrow f'(x) = n x^{n-1}$

Recall: By basic properties of limits (and hence of derivatives), we have that the following hold:

1) Linearity:

$$\lim_{x \to x_0} \left(a f(x) + b g(x)\right) = a \lim_{x \to x_0} f(x) + b \lim_{x \to x_0} g(x)$$

$$\left(a f(x) + b g(x)\right)' = a f'(x) + b g'(x)$$

2) Product rule:

$$\lim_{x \to x_0} \left(f(x) \cdot g(x)\right) = \left(\lim_{x \to x_0} f(x)\right) \cdot \left(\lim_{x \to x_0} g(x)\right)$$

$$\left(f(x) \cdot g(x)\right)' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

$$\left(f(x_0) \cdot g(x_0)\right)' = \lim_{x \to x_0} \frac{f(x) \cdot g(x) - f(x_0) \cdot g(x_0)}{x - x_0}$$

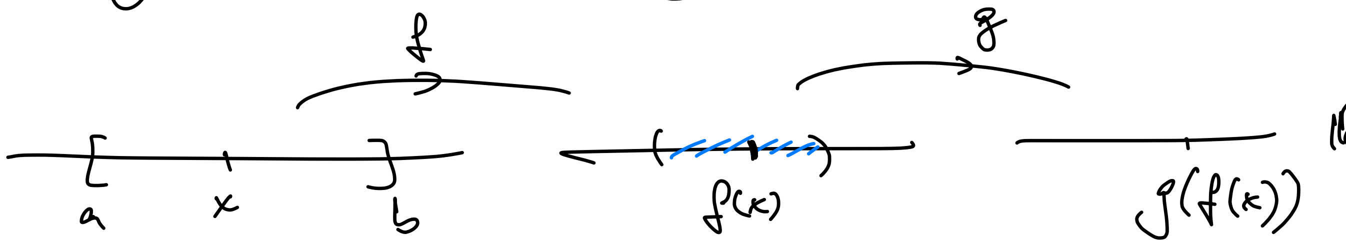
$$= \lim_{x \to x_0} \left(\frac{f(x) \boxed{g(x)} - f(x_0) \boxed{g(x)}}{x - x_0} + \frac{\boxed{f(x_0)} g(x) - \boxed{f(x_0)} g(x_0)}{x - x_0} \right)$$

$$= \underbrace{\lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \right)}_{f'(x_0)} \cdot \underbrace{\lim_{x \to x_0} g(x)}_{g(x_0)} + \underbrace{\lim_{x \to x_0} f(x_0)}_{f(x_0)} \cdot \underbrace{\lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}}_{g'(x_0)}$$

$$= f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0).$$

3) Chain Rule

Prop: Suppose f is cont. on $[a,b]$, $f'(x)$ exists at some $x \in [a,b]$, g is defined on an interval containing $f([a,b])$ and g is differentiable at $f(x)$.



Then $h(t) = g(f(t)) = (g \circ f)(t)$ is differentiable at $t = x$ and $h'(x) = g'(f(x)) \cdot f'(x)$.

Lemma: $\phi(t)$ is differentiable at x if and only if there exist $d \in \mathbb{R}$ and $u(t)$ a function with $\lim_{t \to x} u(t) = 0$ such that $\phi(t) - \phi(x) = (t-x)(d + u(t))$

(In the above, $d = \phi'(x)$)

$$\phi'(x) = \lim_{t \to x} \frac{\phi(t) - \phi(x)}{t - x} = \lim_{t \to x} (d + u(t))$$

$$= d + \underbrace{\lim_{t \to x} u(t)}_{0} = d.$$

Pf (Chain Rule). Since f is diff. at x, by Lemma

$$f(t) - f(x) = (t-x) \cdot (f'(x) + u(t))$$

where $\lim_{t \to x} u(t) = 0.$

Since g is diff at $y = f(x)$, also by Lemma:

$$g(s) - g(y) = (s-y)(g'(y) + v(s))$$

where $\lim_{s \to y} v(s) = 0$. Let $h(t) = g(f(t))$. Then

$$\begin{aligned} h(t) - h(x) &= g(f(t)) - g(f(x)) \\ &= (f(t) - f(x))(g'(f(x)) + v(s)) \\ &= (t-x)\cdot(f'(x) + u(t))\cdot(g'(f(x)) + v(s)) \end{aligned}$$

So, if $t \neq x$, dividing by $t-x$, we find

$$\frac{h(t)-h(x)}{t-x} = (f'(x) + \underbrace{u(t)}_{\substack{\downarrow \\ 0}\ t \to x}) \cdot (g'(f(x)) + \underbrace{v(s)}_{\substack{\downarrow \\ 0}\ s \to y})$$

Letting $t \to x$ and $s \to y$ in the above, we conclude:

$$\underbrace{\lim_{t \to x} \frac{h(t) - h(x)}{t - x}}_{h'(x)} = (f'(x) + 0)(g'(f(x)) + 0) = g'(f(x)) \cdot f'(x).$$

$\square$