Exercise Session

1) Let $f(x)=|x|$ for $x\in\mathbb{R}$. Is there a power series $\sum a_n x^n$ such that $f(x)=\sum a_n x^n$ for all $x\in(-1,1)$?

2) Use the fact that $\underbrace{e^x=\sum_{n=0}^{+\infty}\frac{x^n}{n!} \text{ for all } x\in\mathbb{R}}$ to:

Proved in Lecture 21

a) Express $f(x)=e^{-x^2}$ as a power series centered at $x_0=0$.

b) Express $F(x)=\int_0^x e^{-t^2}\,dt$ as a power series centered at $x_0=0$.

3) Prove that $|\cos x-\cos y|\le|x-y|$ for all $x,y\in\mathbb{R}$.

4) Suppose $f:\mathbb{R}\to\mathbb{R}$ satisfies $|f(x)-f(y)|\le(x-y)^2$ for all $x,y\in\mathbb{R}$. Prove that f is a constant function.

5) Prove that if $f(x)$ is differentiable at $x=x_0$, then

$$f'(x_0)=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h}$$

6) HW 4 a), d)

a) $\sum_{n=1}^{+\infty} 3n^2e^{-n^3}$ $\rightarrow$ Integral test (see solutions!)

$\rightarrow$ Ratio test

$\rightarrow$ Root test

$$\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1 \implies \sum a_n \text{ converges. (absolutely)}$$

$$a_n = 3n^2 \cdot e^{-n^3}, \qquad a_{n+1} = 3(n+1)^2 \cdot e^{-(n+1)^3}$$

$$(n+1)^3 = n^3+3n^2+3n+1$$

$$\left|\frac{a_{n+1}}{a_n}\right| \overset{(>0)}{=} \frac{\cancel{3}(n+1)^2 \cdot e^{-(n+1)^3}}{\cancel{3}n^2 \cdot e^{-n^3}} = \left(\frac{n+1}{n}\right)^2 e^{-(n+1)^3+n^3}$$

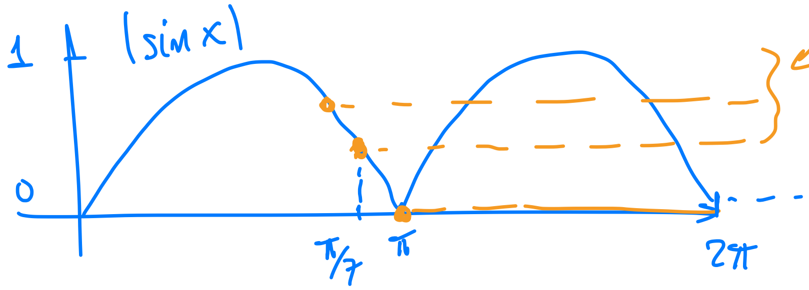
$$= \underbrace{\left(1+\frac{1}{n}\right)^2}_{\to 1} \cdot \underbrace{e^{-3n^2-3n-1}}_{\to 0} \xrightarrow{n \nearrow +\infty} 0. < 1$$

Absolutely b/c $|a_n| = a_n > 0$.

d) $\displaystyle\sum_{n=1}^{+\infty}\left[\sin\left(\frac{n\pi}{7}\right)\right]^n$ ← suggests using Root test

$$\limsup_{n\to\infty}|a_n|^{1/n} = \limsup_{n\to\infty}\left|\sin\left(\frac{n\pi}{7}\right)\right| = \text{"largest subsequential limit of } \left(\left|\sin\left(\tfrac{n\pi}{7}\right)\right|\right)_{n\geq 1}\text{"}$$

$\left(\left|\sin\left(\frac{n\pi}{7}\right)\right|\right)_{n\geq 1}$ $\qquad \lim_{n\to\infty}\left|\sin\frac{n\pi}{7}\right|$ D.N.E.



There are 4 different subsequential limits:

$0,\ \sin\left(\frac{\pi}{7}\right),\ \cos\left(\frac{3\pi}{14}\right),\ \cos\left(\frac{\pi}{14}\right)$

Thus, $\displaystyle\limsup_{n\to\infty}\left|\sin\left(\frac{n\pi}{7}\right)\right| = \cos\left(\frac{\pi}{14}\right) < 1$

Root test $\Longrightarrow$ Series converges absolutely.

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2) Use the fact that $\underbrace{e^x = \sum_{n=0}^{+\infty}\frac{x^n}{n!} \text{ for all } x\in\mathbb{R}}$ to:

Proved in Lecture 21

a) Express $f(x)=e^{-x^2}$ as a power series centered at $x_0=0$.

b) Express $F(x)=\int_0^x e^{-t^2}\,dt$ as a power series centered at $x_0=0$.

a) "$e^y = \sum_{n=0}^{+\infty}\frac{y^n}{n!}$ and set $y=-x^2$"

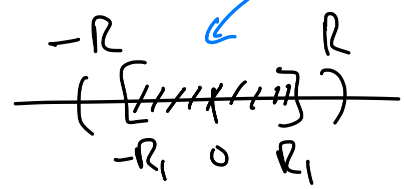
$$f(x)=e^{(-x^2)} = \sum_{n=0}^{+\infty}\frac{(-x^2)^n}{n!} = \sum_{n=0}^{+\infty}\frac{(-1)^n x^{2n}}{n!}$$

Radius of convergence: $R=+\infty$

b)

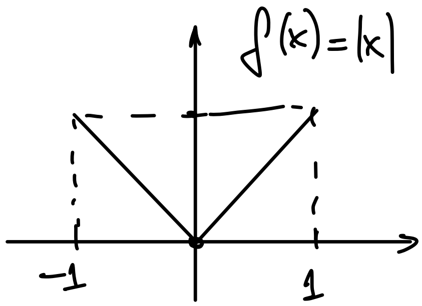
$$F(x)=\int_0^x f(t)\,dt \overset{(a)}{=} \int_0^x\left(\sum_{n=0}^{+\infty}\frac{(-1)^n t^{2n}}{n!}\right)dt \overset{\text{(Lecture 18)}}{=} \sum_{n=0}^{+\infty}\underbrace{\int_0^x \frac{(-1)^n t^{2n}}{n!}\,dt}_{=\frac{(-1)^n}{n!}\cdot\left.\frac{t^{2n+1}}{2n+1}\right|_0^x}$$

$$= \sum_{n=0}^{+\infty}\frac{(-1)^n\cdot x^{2n+1}}{n!\,(2n+1)}$$



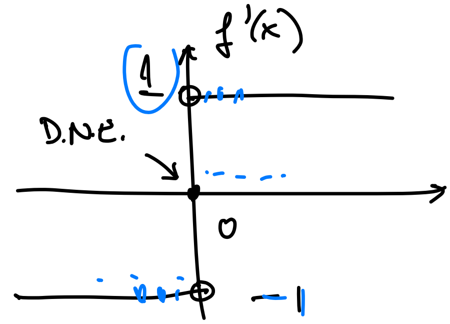
convergence is uniform in this closed interval, hence can exchange integral & limits here. (Lecture 18).

1) Let $f(x)=|x|$ for $x \in \mathbb{R}$. Is there a power series $\sum a_n x^n$ such that $f(x) = \sum a_n x^n$ for all $x \in (-1,1)$?



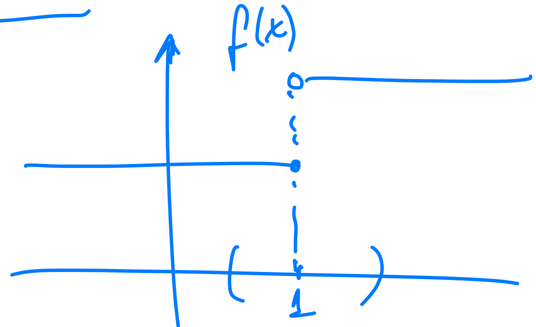
Recall $f(x)=|x|$ is not differentiable at $x_0=0$:

$$f'(x) = \begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases}$$



However, a power series $\sum a_n x^n$ with radius of convergence $R>0$ is always differentiable in $(-R,R)$, with derivative $\sum n \cdot a_n \cdot x^{n-1}$. Therefore $f(x)=|x|$ cannot be written as $\sum a_n x^n$ in any neighborhood of $x_0=0$, because it would then be differentiable at $x_0=0$.

Remark:



$$f(x) \ \text{X} \ \underbrace{\sum_{n=0}^{+\infty} a_n (x-1)^n}$$

not continuous at $x_0=1$

is continuous at $x_0=1$. (even differentiable!)

Thm: (f_n) cont., $f_n \xrightarrow[\text{unif.}]{} f \implies f$ is cont.

4) Suppose $f:\mathbb{R}\to\mathbb{R}$ satisfies $|f(x)-f(y)| \overset{*}{\le} (x-y)^2$ for all $x,y\in\mathbb{R}$. Prove that f is a constant function.

$$|f'(x_0)| = \lim_{x\to x_0}\left|\frac{f(x)-f(x_0)}{x-x_0}\right| = \lim_{x\to x_0}\frac{|f(x)-f(x_0)|}{|x-x_0|} \le$$

$$\overset{*}{\le} \lim_{x\to x_0}\frac{(x-x_0)^2}{|x-x_0|} = \lim_{x\to x_0}\frac{|x-x_0|^{\cancel{2}}}{\cancel{|x-x_0|}} = 0.$$

So $f'\equiv 0$ and hence $f(x)=$ constant. $\square$

5) Prove that if $f(x)$ is differentiable at $x=x_0$, then

$$f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h}$$

<u>Def:</u> $f'(x_0) = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$

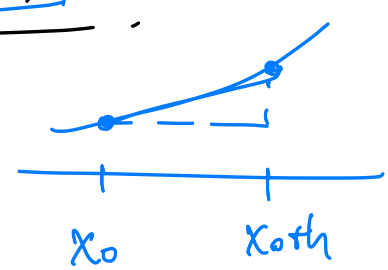
<u>Lemma:</u> $f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$

<u>Pf:</u> Use substitution $x = x_0+h$: $\quad x\to x_0 \Leftrightarrow h\to 0$

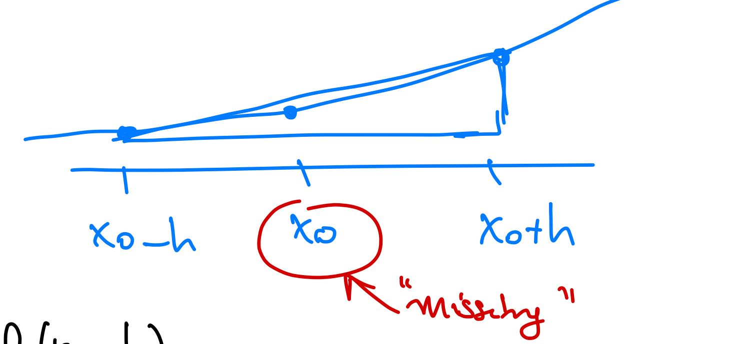
$h = x-x_0$

"Forward difference"

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = \lim_{h\to 0}\frac{\boxed{f(x_0+h)-f(x_0)}}{h}.$$



$$\lim_{h\to 0} \frac{\overset{\text{(forward)}}{f(x_0+h)} - \overset{\text{(backward)}}{f(x_0-h)}}{2h}$$



$$\parallel$$

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)+f(x_0)-f(x_0-h)}{2h} =$$

$$= \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{2h} + \frac{f(x_0)-f(x_0-h)}{2h}$$

$$= \frac{1}{2} \underbrace{\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}}_{f'(x_0)} + \frac{1}{2} \lim_{h\to 0} \frac{f(x_0)-f(x_0-h)}{h}$$

Use substitution

$y = x_0-h, \quad y+h = x_0$

$$\cdots = \lim_{h\to 0} \frac{f(y+h)-f(y)}{h} = f'(x_0)$$

$$= \frac{1}{2} f'(x_0) + \frac{1}{2} f'(x_0)$$

$$= f'(x_0).$$

3) Prove that $|\cos x - \cos y| \leq |x-y|$ for all $x,y \in \mathbb{R}$.

Let $f:\mathbb{R}\to\mathbb{R}$ be given by $f(x) = \cos x$. Recall that $f(x)$ is differentiable at every $x\in\mathbb{R}$ and $f'(x) = -\sin x$.

Given $x,y\in\mathbb{R}$, say $y<x$, we apply the

Mean Value Theorem to $f:[y,x]\to\mathbb{R}$, and obtain:

$$f(x)-f(y) = f'(z)\cdot(x-y)$$

for some $z\in(y,x)$, that is:

$$\cos x - \cos y = (-\sin z)(x-y)$$

Taking absolute values, we find:

$$|\cos x - \cos y| = \underbrace{|-\sin z|}_{\leq 1}\cdot|x-y| \leq |x-y|.$$

$\square$