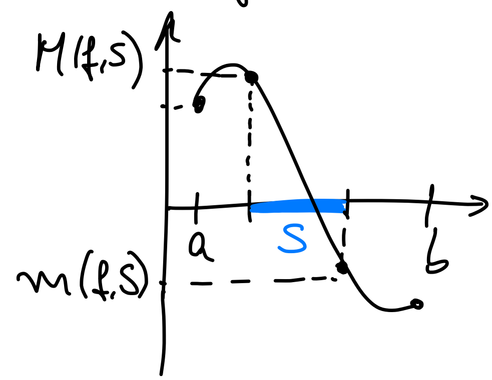
MAT 320/640 Lecture 23 11/24/2021

Riemann Integral

Let $f:[a,b]\to\mathbb{R}$ be a bounded function, and, given $S\subset[a,b]$, define:

$$M(f,S)=\sup\{f(x): x\in S\}$$

$$m(f,S)=\inf\{f(x): x\in S\}$$



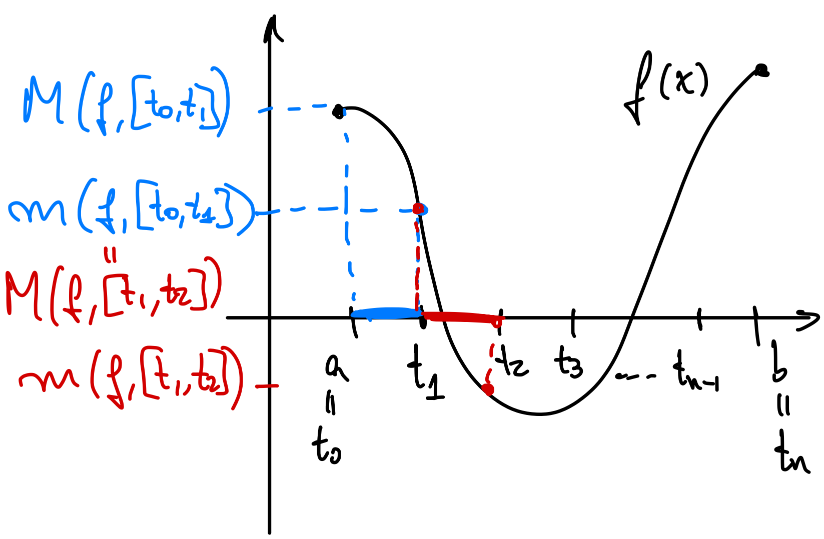
Given a partition P of $[a,b]$, that is,

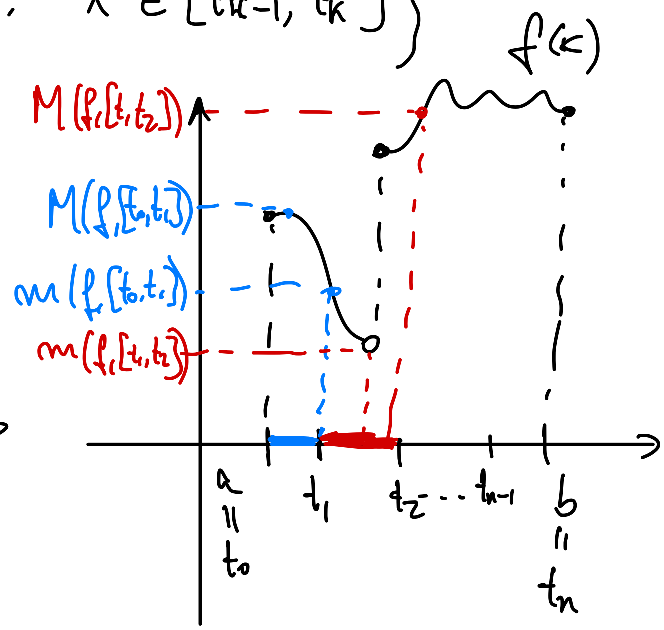
$$P=\{a=t_0<t_1<t_2<\dots<t_n=b\}$$

we have the extremal values

$$M(f,[t_{k-1},t_k])=\sup\{f(x): x\in[t_{k-1},t_k]\}$$

$$m(f,[t_{k-1},t_k])=\inf\{f(x): x\in[t_{k-1},t_k]\}$$

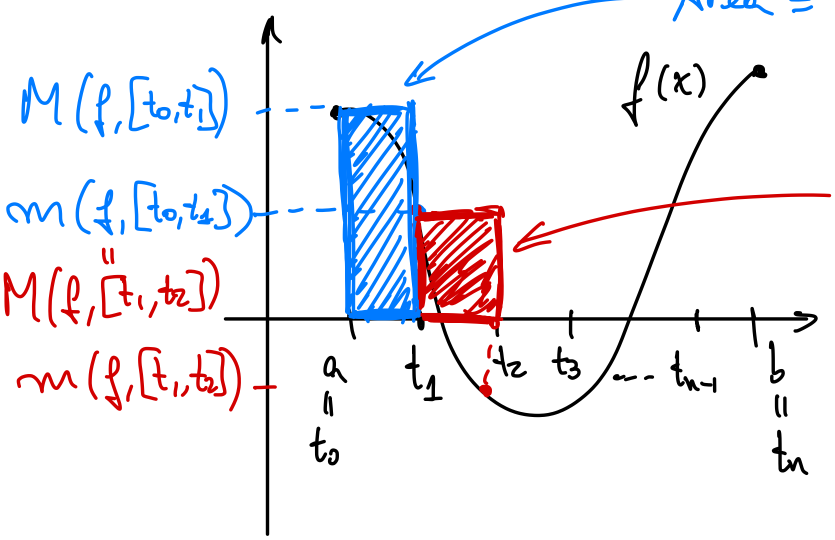




__Definition:__ In the situation above, we define the __upper sum__ of f with respect to the partition P to be

$$U(f,P) := \sum_{k=1}^{n} \underbrace{M(f,[t_{k-1},t_k])}_{\text{height}} \cdot \underbrace{(t_k - t_{k-1})}_{\text{base}}$$

"Area" $= M(f,[t_0,t_1])\cdot(t_1-t_0)$ $(k=1)$

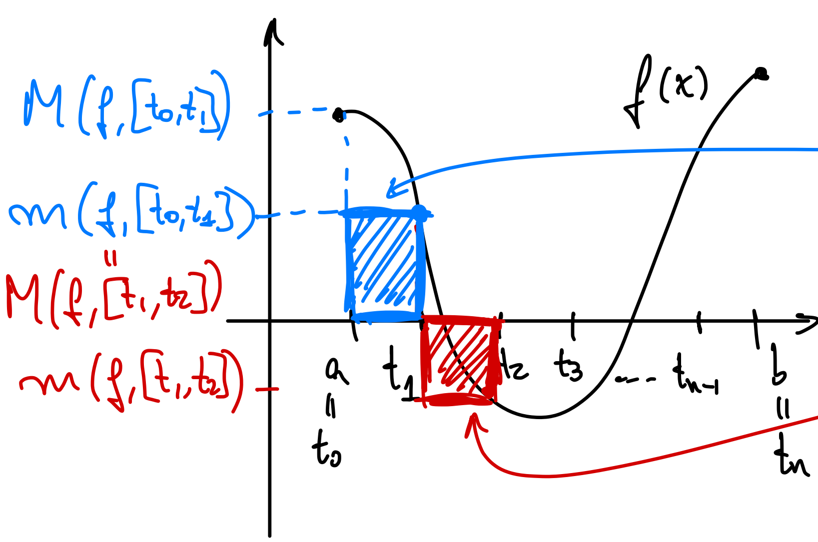


"Area" $= M(f,[t_1,t_2])\cdot(t_2-t_1)$

$(k=2)$

and the __lower sum__ of f w.r.t. the partition P as:

$$L(f,P) = \sum_{k=1}^{n} \underbrace{m(f,[t_{k-1},t_k])}_{\text{height}} \cdot \underbrace{(t_k - t_{k-1})}_{\text{base}}$$



"Area" $= m(f,[t_0,t_1])\cdot(t_1-t_0)$

"Area" $= m(f,[t_1,t_2])(t_2-t_1)$

(signed)

<u>Definition</u>: In the notation above, we define the <u>Upper integral</u> of f on $[a,b]$ to be:

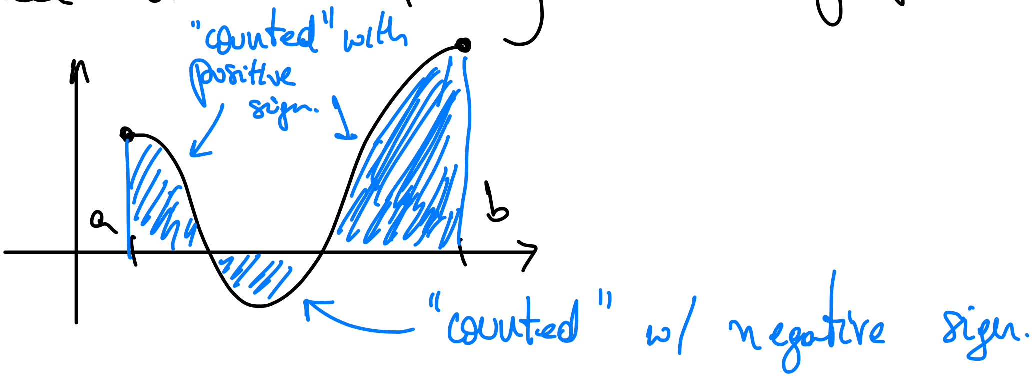
$$U(f) = \overline{\int_a^b} f = \inf\{U(f,P) : P \text{ partition of } [a,b]\}.$$

and the <u>lower integral</u> of f on $[a,b]$ to be:

$$L(f) = \underline{\int_a^b} f = \sup\{L(f,P) : P \text{ partition of } [a,b]\}.$$

<u>Definition</u>. The function $f : [a,b] \to \mathbb{R}$ is (Darboux) integrable on $[a,b]$ if $U(f) = L(f)$. In that case, we write $\int_a^b f = \int_a^b f(x)\,dx = U(f) = L(f)$.

As indicated above, $U(f,P)$ and $L(f,P)$ are sums of (signed) areas of rectangles whose base is an interval between adjacent points of the partition P. Taking inf & sup as above, we obtain "limiting" signed area determined by the graph of $f : [a,b] \to \mathbb{R}$.

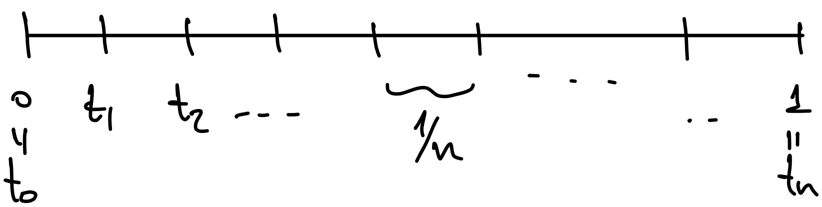


In the sequel, we prove that $L(f) \leq U(f)$. For now, let us use that fact to work through some examples.

<u>Examples.</u>

1) $f:[0,1]\to\mathbb{R}$, $\quad f(x)=x^2$ $\quad$ (We expect: $\displaystyle\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$.)

Consider a partition of $[0,1]$ into subintervals of equal size:



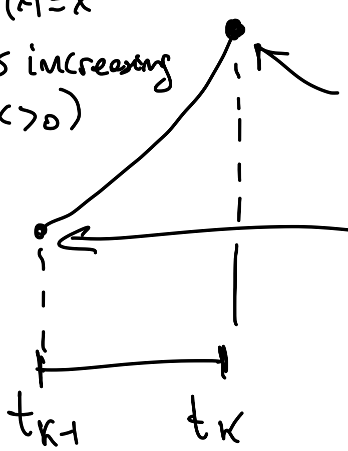
$t_0 = 0,\ t_1 = \frac{1}{n},\ t_2 = \frac{2}{n}, \ldots$

$$\boxed{t_k = \frac{k}{n}, \quad k = 0,\ldots,n}$$

$$P_n = \left\{ t_k = \frac{k}{n},\ k = 0,1,\ldots,n \right\}$$

$$U(f,P_n) = \sum_{k=1}^{n} \underbrace{M(f,[t_{k-1},t_k])}_{\left(\frac{k}{n}\right)^2} \cdot \underbrace{(t_k - t_{k-1})}_{\frac{k}{n}-\frac{k-1}{n} = \frac{1}{n}(k-(k-1)) = \boxed{\frac{1}{n}}} = \sum_{k=1}^{n}\left(\frac{k}{n}\right)^2\cdot\frac{1}{n} = \ldots$$

$f(x)=x^2$ is increasing $(x>0)$



$$M(f,[t_{k-1},t_k]) = f(t_k) = t_k^2 = \left(\frac{k}{n}\right)^2$$

$$m(f,[t_{k-1},t_k]) = f(t_{k-1}) = t_{k-1}^2 = \left(\frac{k-1}{n}\right)^2.$$

$$1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\ldots = \frac{1}{n^3}\sum_{k=1}^{n} k^2 = \frac{1}{n^3}\left(1^2+2^2+3^2+\cdots+n^2\right) = \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

Note $$\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} \frac{2n^3+3n^2+n}{6n^3}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^3}\left(2 + \overset{\to 0}{\frac{3}{n}} + \overset{\to 0}{\frac{1}{n^2}}\right)}{6\cancel{n^3}} = \frac{2}{6} = \frac{1}{3}.$$

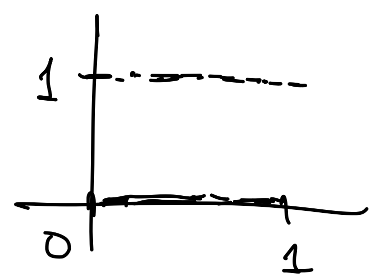
Similarly, $\lim_{n\to\infty} L(f, P_n) = \frac{1}{3}$.

In fact, $U(f) \le \frac{1}{3}$ and $L(f) \geq \frac{1}{3}$, therefore

$$\frac{1}{3} \le L(f) \overset{\text{Proved later}}{\le} U(f) \le \frac{1}{3}$$ hence $L(f) = U(f)$, i.e.

$f(x) = x^2$ is integrable on $[0,1]$ and $\int_0^1 x^2 dx = \frac{1}{3}$.

2) $f : [0,1] \to \mathbb{R}$, $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$



Claim: f is not (Darboux) integrable.

Since in every interval $[t_{k-1}, t_k]$ with $t_{k-1} < t_k$ there exist both rational and irrational numbers, we have that for any partition $P = \{0 = t_0 < t_1 < \dots < t_n = 1\}$

$$L(f, P) = \sum_{k=1}^{n} \underbrace{m(f, [t_{k-1}, t_k])}_{=0,\ \forall k.} \cdot (t_k - t_{k-1}) = 0$$

$$U(f,P) = \sum_{k=1}^{n} \underbrace{M(f,[t_{k-1},t_k])}_{=1,\ \forall k} \cdot (t_k - t_{k-1}) = \sum_{k=1}^{n} t_k - t_{k-1}$$

$$= (\cancel{t_1} - t_0) + (\cancel{t_2} - \cancel{t_1}) + (\cancel{t_3} - \cancel{t_2}) + \cancel{-} \cancel{-} + (t_n - \cancel{t_{n-1}})$$

"telescopic sum"

$$= t_n - t_0 = 1 - 0 = 1.$$

Therefore,

$$L(f) = \sup\{\underbrace{L(f,P)}_{=0} : P \text{ partition of } [0,1]\} = 0$$

$$\wedge$$

$$U(f) = \inf\{\underbrace{U(f,P)}_{=1} : P \text{ partition of } [0,1]\} = 1$$

Thus $L(f) \neq U(f)$ and hence f is not integrable.

Lemma. Let $f:[a,b]\to\mathbb{R}$ be bounded and P_1 and P_2 be partitions of $[a,b]$, with $P_1 \subset P_2$. Then the following holds:

"P_2 refines P_1"

$$L(f,P_1) \le L(f,P_2) \le U(f,P_2) \le U(f,P_1)$$

Pf: See Ross' book (Lemma 32.2, p. 273)

Thm. If $f:[a,b]\to\mathbb{R}$ is bounded, then $L(f) \le U(f)$.

Proof. First, we claim that if P and Q are partitions of $[a,b]$, then $L(f,P) \le U(f,Q)$.

Indeed, $P \cup Q$ is a partition that refines both P and Q, so by Lemma:

PUQ refines Q

$$L(f,P) \le L(f, P\cup Q) \le U(f, P \cup Q) \le U(f,Q).$$

PUQ refines P.

Recall $\quad U(f) = \inf\{ U(f,P) \; : \; P \text{ partition of } [a,b]\}$

$$L(f) = \sup\{ L(f,P) \; : \; P \text{ partition of } [a,b]\}$$

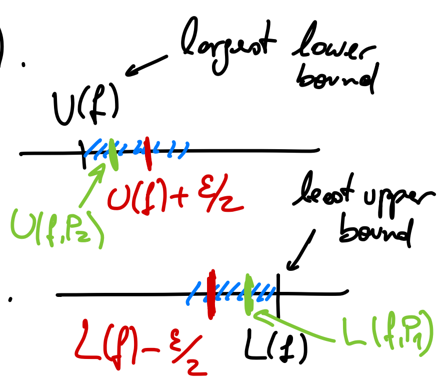
Since $L(f,P) \le U(f,Q)$ taking the supremum as above, we have $L(f) \le U(f,Q)$ for any fixed partition Q of $[a,b]$. Taking an infimum over Q, we find

$$L(f) \le U(f) = \inf\{U(f,Q) : Q \text{ partition of } [a,b]\} \qquad \square$$

<u>Theorem</u>. A bounded function $f : [a,b] \to \mathbb{R}$ is (Darboux) integrable if and only if $\forall \varepsilon > 0$ there exists a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$.

<u>Pf</u>. Suppose f is integrable, i.e., $U(f) = L(f)$.

$$U(f) = \inf\{U(f,P) : P \text{ partitions of } [a,b]\}$$
$$\shortparallel$$
$$L(f) = \sup\{L(f,P) : P \text{ partitions of } [a,b]\}.$$



Given $\varepsilon > 0$, there exist P_1 and P_2 partitions of $[a,b]$ s.t. $L(f,P_1) > L(f) - \frac{\varepsilon}{2}$ & $U(f,P_2) < U(f) + \frac{\varepsilon}{2}$.

Consider $P = P_1 \cup P_2$, which refines both P_1 and P_2 simultaneously. By Lemma, we have:

$$U(f,P) - L(f,P) \leq U(f,P_2) - L(f,P_1) < U(f) + \frac{\varepsilon}{2} - \left(L(f) - \frac{\varepsilon}{2}\right)$$

$U(f,P) \leq U(f,P_2)$

$-L(f,P) \leq -L(f,P_1)$

$(\Leftarrow: L(f,P) \geq L(f,P_1))$

$$= \underbrace{\left[U(f) - L(f)\right]}_{=0 \text{ by hypothesis}} + \varepsilon.$$

Conversely, if $\forall \varepsilon > 0\ \exists P$ partition of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$, then

$$U(f) \leq \underbrace{U(f,P) - L(f,P)}_{< \varepsilon} + L(f,P)$$

$U(f) = \inf \ldots$

$L(f) = \sup \ldots$

$$< \varepsilon + L(f,P) \leq L(f) + \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we conclude from $U(f) < L(f) + \varepsilon$ that $U(f) \leq L(f)$. From previous result, we have $L(f) \leq U(f)$, so we find $U(f) = L(f)$. $\square$