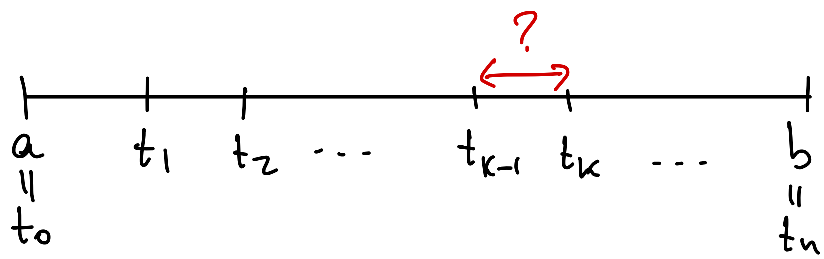
Recall from Lecture 23:

$$P = \{ a = t_0 < t_1 < t_2 < \dots < t_n = b \} \quad \text{partition}$$

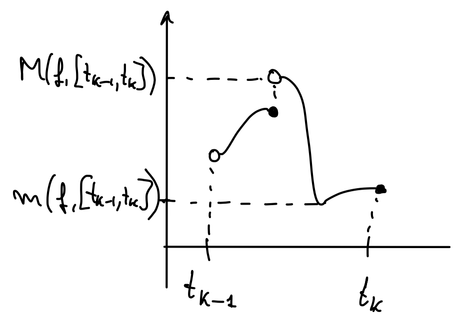


largest distance between consecutive points in P.

<u>Def</u>: $\operatorname{mesh}(P) = \max \{ t_K - t_{K-1} : K = 1, \dots, n \}$

$$M(f, [t_{K-1}, t_K]) = \sup \{ f(x) : x \in [t_{K-1}, t_K] \}$$

$$m(f, [t_{K-1}, t_K]) = \inf \{ f(x) : x \in [t_{K-1}, t_K] \}$$



Upper sum

$$U(f,P) := \sum_{K=1}^{n} M(f, [t_{K-1}, t_K]) \cdot (t_K - t_{K-1})$$

lower sum

$$L(f,P) = \sum_{K=1}^{n} m(f, [t_{K-1}, t_K]) \cdot (t_K - t_{K-1})$$

Upper integral:

$$U(f) = \overline{\int_a^b} f = \inf \{ U(f,P) : P \text{ partition of } [a,b] \}.$$

Lower integral:

$$L(f) = \underline{\int_a^b} f = \sup \{ L(f,P) : P \text{ partition of } [a,b] \}.$$

In general, $L(f) \leq U(f)$.

<u>Def:</u> f is integrable on $[a,b]$ if $U(f) = L(f)$,

and, in that case, $\int_a^b f(x)\,dx = U(f) = L(f)$.

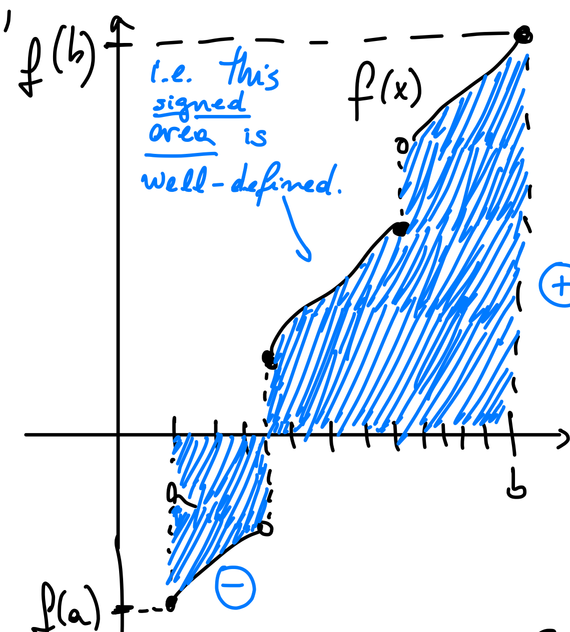
Finally, recall:

<u>Theorem</u>. A bounded function $f:[a,b] \to \mathbb{R}$ is (Darboux) integrable if and only if $\forall \varepsilon > 0$ there exists a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$.

<u>Thm</u>. If $f:[a,b] \to \mathbb{R}$ is monotonic, then f is integrable on $[a,b]$.

<u>Pf</u>: Assume $f:[a,b] \to \mathbb{R}$ is monotonic increasing.

Let $\varepsilon > 0$ be given. Since $f(a) \leq f(x) \leq f(b)$ for all $x \in [a,b]$. Without loss of generality, suppose $f(a) < f(b)$, otherwise f would be constant. We define a partition P of $[a,b]$ with



$$\text{mesh}(P) < \frac{\varepsilon}{f(b) - f(a)}$$

e.g., by refining any given partition until $\text{mesh}(P) < \dfrac{\varepsilon}{f(b) - f(a)}$.

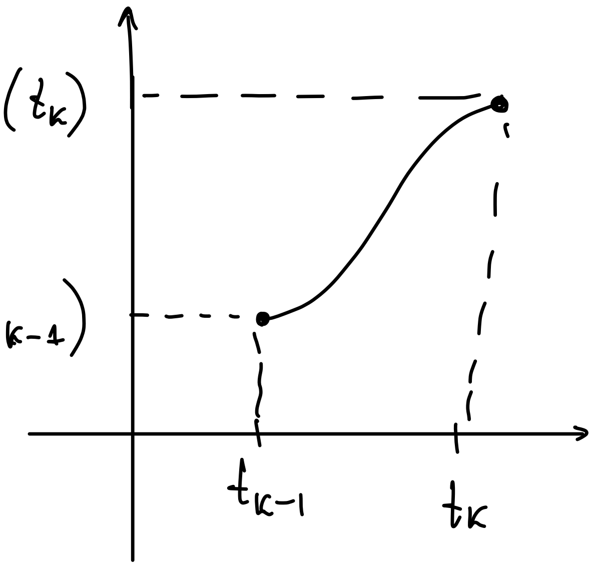
Let us compute $U(f,P) - L(f,P)$; noting that

$$m(f, [t_{k-1}, t_k]) = f(t_{k-1}), \quad M(f, [t_{k-1}, t_k]) = f(t_k)$$

b/c $f(x)$ is increasing (see picture).

$M(f,[t_{k-1},t_k]) = f(t_k)$

$m(f,[t_{k-1},t_k]) = f(t_{k-1})$



$$U(f,P) - L(f,P) = \sum_{k=1}^{n} M(f,[t_{k-1},t_k]) \cdot (t_k - t_{k-1}) - \sum_{k=1}^{n} m(f,[t_{k-1},t_k]) \cdot (t_k - t_{k-1})$$

factor

$$= \sum_{k=1}^{n} \Big(\underbrace{M(f,[t_{k-1},t_k])}_{f(t_k)} - \underbrace{m(f,[t_{k-1},t_k])}_{f(t_{k-1})} \Big)(t_k - t_{k-1})$$

$$= \sum_{k=1}^{n} \big(f(t_k) - f(t_{k-1}) \big)(t_k - t_{k-1})$$

$$\leq \mathrm{mesh}(P) < \frac{\varepsilon}{f(b) - f(a)}$$

$$< \sum_{k=1}^{n} \left(f(t_k) - f(t_{k-1})\right) \cdot \left(\frac{\varepsilon}{f(b)-f(a)} \right)$$

$$= \left(\frac{\varepsilon}{f(b)-f(a)} \right) \cdot \underbrace{\sum_{k=1}^{n} \left(f(t_k) - f(t_{k-1})\right)}$$

$$= \cancel{f(t_1)} - f(t_0) + \cancel{f(t_2)} - \cancel{f(t_1)} + \cancel{\dots}$$

$$\cancel{\dots} + f(t_n) - \cancel{f(t_{n-1})}$$

telescopic

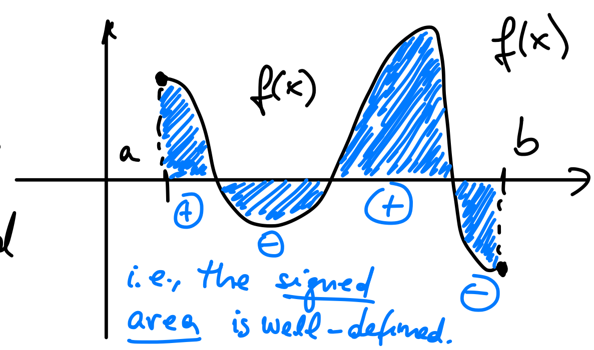
$$= f(t_n) - f(t_0) = f(b) - f(a)$$

$$= \left(\frac{\varepsilon}{\cancel{f(b)-f(a)}} \right) \cdot \left(\cancel{f(b)-f(a)} \right) = \varepsilon.$$

Therefore $U(f,P) - L(f,P) < \varepsilon$, hence f is integrable on $[a,b]$ by the theorem above. $\square$

<u>Thm</u>. If $f : [a,b] \to \mathbb{R}$ is continuous, then f is integrable on $[a,b]$.

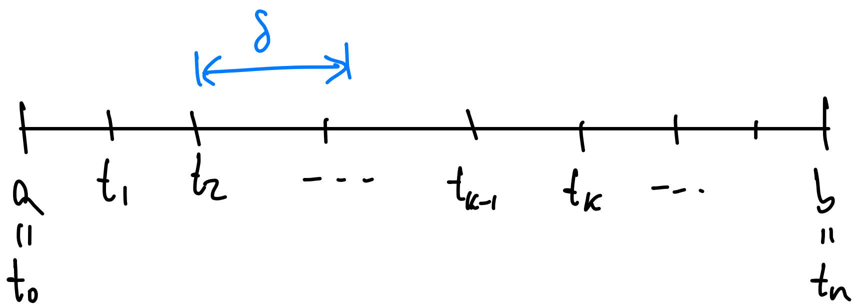
<u>Pf</u>: From results in Lecture 14, $f : [a,b] \to \mathbb{R}$ is <u>uniformly</u> <u>continuous</u> (b/c it is continuous on a closed and bounded interval $[a,b]$.)



Therefore, for any given $\varepsilon > 0$, there exists $\delta > 0$ s.t.

$$\forall x, y \in [a,b], \quad |x-y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{b-a}.$$

Motivated by this, we choose a partition P of $[a,b]$ with $\text{mesh}(P) < \delta$. (This is possible by refining any given partition sufficiently many times).



$\text{mesh}(P) < \delta$.

Since f is continuous, it attains max & min in any closed interval; in particular, on each of $[t_{k-1}, t_k]$, $k = 1, 2, \dots, n$. and:

$$\underbrace{M(f, [t_{k-1}, t_k])}_{= \max\limits_{x \in [t_{k-1}, t_k]} f(x)} - \underbrace{m(f, [t_{k-1}, t_k])}_{= \min\limits_{x \in [t_{k-1}, t_k]} f(x)} < \frac{\varepsilon}{b-a}.$$

Therefore

$$U(f,P) - L(f,P) = \sum_{k=1}^{n} \underbrace{\Big(M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k])\Big)}_{< \varepsilon/(b-a)} (t_k - t_{k-1})$$

$$< \frac{\varepsilon}{b-a} \cdot \underbrace{\sum_{k=1}^{n} (t_k - t_{k-1})}_{\substack{= \cancel{t_1} - t_0 + \cancel{t_2} - \cancel{t_1} + \cancel{\cdots} + t_n - \cancel{t_{n-1}} \\ = -t_0 + t_n = t_n - t_0 = b-a}} = \varepsilon;$$

which implies that $f(x)$ is integrable.

$\square$

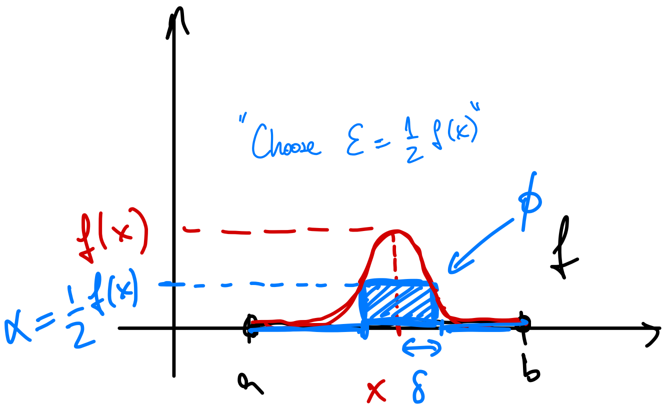
<u>Theorem</u>. Let $f, g : [a,b] \to \mathbb{R}$ be integrable functions. Then:

linearity of integration

- $f+g$ is integrable, and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.
- for any $c \in \mathbb{R}$, the function $c \cdot f$ is integrable, and $\int_a^b c \cdot f = c \cdot \int_a^b f$.
- if $f(x) \leq g(x)$ for all $x \in [a,b]$, then $\int_a^b f \leq \int_a^b g$.
- $|f|$ is integrable, and $\left| \int_a^b f \right| \leq \int_a^b |f|$.

<u>Pf</u>: Read Section 33 of Ross.

<u>Corollary</u>. Suppose $f : [a,b] \to \mathbb{R}$ is continuous and nonnegative, i.e., $f(x) \geq 0$ for all $x \in [a,b]$, and $\int_a^b f = 0$. Then $f \equiv 0$.



$\phi \le f$

<u>Pf:</u> Suppose $\exists\, x \in (a,b)$ such that $f(x) > 0$. Since f is continuous, there exists $\delta > 0$ s.t. $f(t) \geq \frac{f(x)}{2} = \alpha$ for all $|t - x| < \delta$. Then, by Thm above,

$$\int_a^b f \geq \int_a^b \phi = \int_{x-\delta}^{x+\delta} \alpha = \alpha \cdot 2\delta > 0$$

where $\phi \equiv \alpha$ in $[x-\delta, x+\delta]$ and $\phi \equiv 0$ otherwise.
This contradicts $\int_a^b f = 0$, concluding the proof. $\square$

Back to the question:

<u>Q:</u> $$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \underbrace{\left(\lim_{n\to\infty} f_n(x)\right)}_{f(x)} dx \ ?$$

<u>A:</u> In general, <u>NO</u>. But ...

- <u>YES</u>, if $f_n \longrightarrow f$ uniformly and if f_n are continuous. (in this case, $f = \lim_{n\to\infty} f_n$ is also continuous, and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.)

- YES, if $f_n \to f$ uniformly and if f_n are integrable (in this case, $f = \lim_{n\to\infty} f_n$ is also integrable, and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.)

What if f_n does not converge uniformly to f?

If $f_n \to f$ only pointwise, the equality above may fail. For example, take

$$f : [0,1] \to \mathbb{R}, \quad f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

and recall we proved that $f(x)$ is not integrable on $[0,1]$, since $L(f) = 0 < 1 = U(f)$.

Let $(x_k)_{k\in\mathbb{N}}$ be an enumeration of $\mathbb{Q} \cap [0,1]$, and let $f_n : [0,1] \to \mathbb{R}$ be the functions defined by

$$f_n(x) = \begin{cases} 1 & \text{if } x = x_k \text{ for some } k = 1, \dots, n \\ 0 & \text{otherwise} \end{cases}$$

Clearly, $f_n \to f$ pointwise (but not uniformly);

and $\int_0^1 f_n \, dx = 0$.

Even with pointwise convergence, there are situations in which we still have an affirmative answer, assuming that $f = \lim f_n$ is integrable.

This is b/c we are working with Riemann / Darboux integrability.

Dominated Convergence Theorem

Suppose $f_n : [a,b] \to \mathbb{R}$ are (Riemann) integrable, and $f_n \to f$ pointwise and f is also (Riemann) integrable. If there exists $M > 0$ such that $|f_n(x)| \le M$ for all $x \in [a,b]$ and $n \in \mathbb{N}$, then

"Dominated convergence"

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \underbrace{\lim_{n\to\infty} f_n(x)}_{f(x)}\,dx.$$