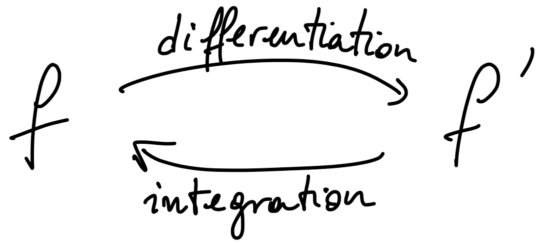
MAT 320/640 Lecture 25 12/1/2021



Fundamental Theorem of Calculus I. Let $f:[a,b]\to\mathbb{R}$ be differentiable on (a,b), with f' integrable on $[a,b]$. Then:

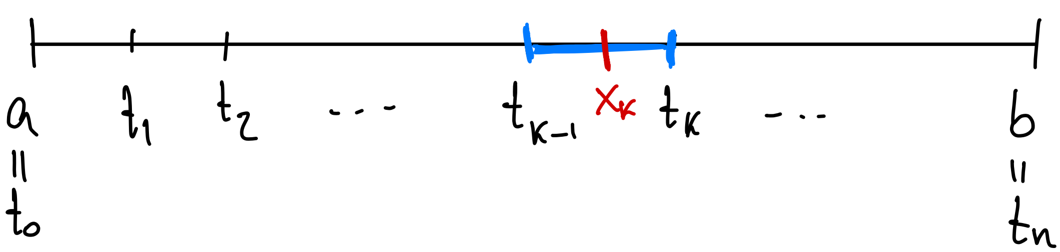
$$\int_a^b f' = f(b) - f(a)$$

This is how we compute definite integrals in Calculus

Proof. Recall f' is integrable if and only if $\forall \varepsilon > 0$, there is a partition

$$P = \{a = t_0 < t_1 < t_2 < \dots < t_n = b\}$$

of $[a,b]$ s.t. $U(f',P) - L(f',P) < \varepsilon$.



Applying the Mean Value Theorem to f on each interval $[t_{k-1}, t_k]$, we find $\exists x_k \in (t_{k-1}, t_k)$

$$f(t_k) - f(t_{k-1}) = f'(x_k)(t_k - t_{k-1})$$

$$\sum_{k=1}^{n} f'(x_k)(t_k - t_{k-1}) = \sum_{k=1}^{n} f(t_k) - f(t_{k-1})$$

$$\overset{\text{telescopic}}{=} f(t_n) - f(t_0) = f(b) - f(a)$$

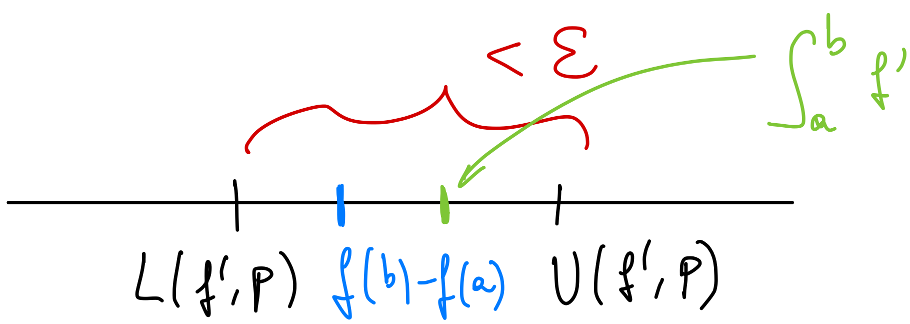
Comparing the above with upper/lower sums for f', we find:

$$\underbrace{\sum_{k=1}^{n} m(f',[t_{k-1},t_k])(t_k - t_{k-1})}_{L(f',P)} \le \sum_{k=1}^{n} \underbrace{f'(x_k)}_{m(f',[t_{k-1},t_k]) \le \dots \le M(f',[t_{k-1},t_k])}(t_k - t_{k-1}) \le \underbrace{\sum_{k=1}^{n} M(f',[t_{k-1},t_k]) \cdot (t_k - t_{k-1})}_{U(f',P)}$$

$$L(f',P) \le \underbrace{\sum_{k=1}^{n} f'(x_k)(t_k - t_{k-1})}_{= f(b)-f(a)} \le U(f',P)$$

Moreover, also $\int_a^b f'$ is in that interval:

$$L(f',P) \le \int_a^b f' \le U(f',P)$$



So $$\left| \left(\int_a^b f'\right) - \left(f(b) - f(a)\right) \right| < \varepsilon .$$

can be made arbitrarily small

Thus, $$\int_a^b f' = f(b) - f(a).$$

□

<u>Theorem</u> (Integration by Parts). If $u, v : [a,b] \to \mathbb{R}$ are differentiable and u', v' are continuous, then

it is actually enough to assume they are integrable.

$$\int_a^b u'v = uv\Big|_a^b - \int_a^b uv'$$

<u>Pf</u>: Let $f = uv$. Then $f' = u'v + uv'$. By F.T.C.:

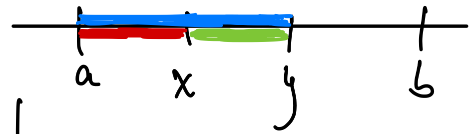
$$\int_a^b u'v + uv' = \int_a^b f' = f(b) - f(a) = u \cdot v \Big|_a^b$$

$$\int_a^b u'v + \int_a^b uv' \qquad \Longrightarrow \int_a^b u'v = uv\Big|_a^b - \int_a^b uv'.$$

□

<u>Fundamental Theorem of Calculus II</u>. Let $f : [a,b] \to \mathbb{R}$ be integrable, and let $F(x) = \int_a^x f(t)\,dt$. Then $F(x)$ is continuous at every $x_0 \in [a,b]$, and, if $f(x)$ is continuous at x_0, then $F(x)$ is differentiable at x_0 with $F'(x_0) = f(x_0)$.

Pf: Since f is integrable on $[a,b]$, there exists $B>0$ s.t. $|f(x)| \le B$ for all $x \in [a,b]$. Given $\varepsilon > 0$, let $\delta = \varepsilon / B$. Then if $x, y \in [a,b]$ s.t. $|x-y| < \delta$, then (assume WLOG that $x<y$)



$$\left| F(y) - F(x) \right| = \left| \int_a^y f(t)\,dt - \int_a^x f(t)\,dt \right|$$

$|x-y| < \delta = \frac{\varepsilon}{B}$

$$= \left| \int_x^y f(t)\,dt \right| \le \int_x^y \underbrace{|f(t)|}_{\le B}\,dt \le B \cdot (y-x) < B \cdot \frac{\varepsilon}{B} = \varepsilon.$$

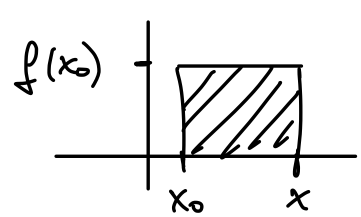
Therefore F is (uniformly) continuous on $[a,b]$.

Note that, for all $x \neq x_0$:

w/ convention $\int_a^b f = -\int_b^a f$

$$\frac{F(x) - F(x_0)}{x - x_0} = \frac{\int_a^x f(t)\,dt - \int_a^{x_0} f(t)\,dt}{x - x_0} = \frac{1}{x - x_0} \cdot \left(\int_{x_0}^x f(t)\,dt \right)$$

and $\displaystyle\int_{x_0}^x f(x_0)\,dt = f(x_0) \cdot \int_{x_0}^x dt = f(x_0) \cdot (x - x_0)$



thus, if $x \neq x_0$, we have

$$\frac{1}{x - x_0} \int_{x_0}^x f(x_0)\,dt = f(x_0).$$

Therefore

$$\frac{1}{x-x_0}\int_{x_0}^{x}\big(f(t)-f(x_0)\big)\,dt = \underbrace{\frac{1}{x-x_0}\int_{x_0}^{x} f(t)\,dt}_{} - \underbrace{\frac{1}{x-x_0}\int_{x_0}^{x} f(x_0)\,dt}_{}$$

$$= \frac{F(x)-F(x_0)}{x-x_0} - f(x_0)$$

If $f(t)$ is continuous at $t=x_0$, then $\forall \varepsilon>0$ there exists $\delta>0$ s.t. $|t-x_0|<\delta$ then $|f(t)-f(x_0)|<\varepsilon$.

So, if $|x-x_0|<\delta$, then $|t-x_0|<\delta$ for all $t\in(x_0,x)$,

so

$$\left|\frac{F(x)-F(x_0)}{x-x_0}-f(x_0)\right| = \left|\frac{1}{x-x_0}\int_{x_0}^{x}\big(f(t)-f(x_0)\big)\,dt\right|$$

$$\leq \frac{1}{x-x_0}\int_{x_0}^{x}\underbrace{|f(t)-f(x_0)|}_{<\varepsilon}\,dt$$

$$< \frac{1}{x-x_0}\int_{x_0}^{x}\varepsilon\,dt = \frac{\varepsilon\cdot\cancel{(x-x_0)}}{\cancel{x-x_0}}$$

$$= \varepsilon.$$

Can be made arbitrarily small!

Therefore $$\lim_{x\to x_0} \frac{F(x)-F(x_0)}{x-x_0} - f(x_0) = 0$$

i.e. $$F'(x_0) = \lim_{x\to x_0} \frac{F(x)-F(x_0)}{x-x_0} = f(x_0). \qquad \square$$

<u>Thm</u> (Integration by substitution). Let $u:[a,b]\to[c,d]$ be a differentiable function, with u' continuous and $u(a)=c$, $u(b)=d$. If $f:[c,d]\to\mathbb{R}$ is continuous then $f\circ u$ is continuous and:

$$\int_a^b \underbrace{f(u(x))}_{f(y)}\,\underbrace{u'(x)\,dx}_{dy} = \int_c^d f(y)\,dy$$

"u-substitution"
$y = u(x)$
$dy = u'(x)\,dx$

<u>Pf</u>: Composition of continuous functions are continuous, so $(f\circ u)$ is continuous. Fix $y_0\in[c,d]$ and define

$$F(y) = \int_{y_0}^{y} f(t)\,dt$$

By F.T.C. II, $F'(y)=f(y)$. Let $g = F\circ u$, i.e., $g(x) = F(u(x)) = \int_{y_0}^{u(x)} f(t)\,dt$. By the Chain Rule,

$$g'(x) = \underbrace{F'(u(x))}_{= f(u(x))} \cdot u'(x) = f(u(x)) \cdot u'(x)$$

Integrating both sides:

F.T.C. I

$$\int_a^b f(u(x))\,u'(x)\,dx = \int_a^b g'(x)\,dx \overset{\text{F.T.C. I}}{=} g(b) - g(a)$$

$$= \int_{y_0}^{u(b)} f(t)\,dt - \int_{y_0}^{u(a)} f(t)\,dt = \int_{u(a)=c}^{u(b)=d} f(t)\,dt = \int_c^d f(y)\,dy.$$

$\square$