Sequences

Definition: A sequence (of real numbers) is a function $s: \{n \in \mathbb{Z} : n \geq m\} \to \mathbb{R}$, where $m \in \mathbb{Z}$ is given.

(Typically, $m = 0$ or $m = 1$)

Example: $m = 1$, $s(n) = \frac{1}{n^2}$

$$s : \underbrace{\{1, 2, 3, \dots\}}_{\mathbb{N}} \to \mathbb{R}$$

$s(1) = 1$

$s(2) = \frac{1}{2^2} = \frac{1}{4}$

$s(3) = \frac{1}{3^2} = \frac{1}{9}$

$\vdots$

Other ways to write the sequence:

- $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \dots$
- $\boxed{(s_n)_{n\in\mathbb{N}}, \quad s_n = \frac{1}{n^2}.}$

We will mostly use this notation.

Example: $\boxed{(a_n)_{n \geq 0}, \quad a_n = (-1)^n}$

$m = 0$

$a_0 = (-1)^0 = 1$

$a_1 = (-1)^1 = -1$

$a_2 = (-1)^2 = 1$

$\vdots$

$1, -1, 1, -1, 1, \dots$

$\left(\begin{array}{l} a : \{0, 1, 2, 3, \dots\} \to \mathbb{R} \\ a(n) = (-1)^n \end{array} \right)$

$$a_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd.} \end{cases}$$

Example: $\boxed{(b_n)_{n\in\mathbb{N}}, \quad b_n = \sqrt[n]{n}}$

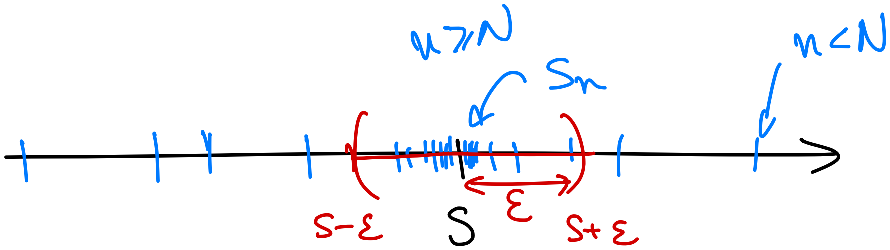
$b : \{1, 2, 3, \dots\} \to \mathbb{R}$

$b(n) = \sqrt[n]{n} = (n)^{1/n}$

$1, \sqrt[2]{2}, \sqrt[3]{3}, \sqrt[4]{4}, \dots$

Limit of a Sequence

<u>Definition</u>: A sequence (s_n) of real numbers <u>converges</u> to a limit $s \in \mathbb{R}$ if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $\underbrace{|s_n - s| < \varepsilon}_{s_n \in (s-\varepsilon,\, s+\varepsilon)}$.



"No matter how small $\varepsilon > 0$ is, there is $N \in \mathbb{N}$ (depending on ε) such that all elements of the sequence "after N" lie within distance ε from s!"

<u>Notation</u>: $\lim\limits_{n \to \infty} s_n = s$, or $s_n \to s$.

<u>Example</u>: $\lim\limits_{n \to \infty} \dfrac{3n+1}{7n-4} = \,?$

$(s_n)_{n \in \mathbb{N}}$, $s_n = \dfrac{3n+1}{7n-4}$

$\boxed{\text{From Calculus, we are inclined to assume this limit will be } \frac{3}{7}.}$

$$\lim_{n \to \infty} \frac{3n+1}{7n-4} = \lim_{n \to \infty} \frac{\cancel{n}\left(3 + \frac{1}{n}\right)}{\cancel{n}\left(7 - \frac{4}{n}\right)} = \lim_{n \to \infty} \frac{3 + \frac{1}{n}}{7 - \frac{4}{n}}$$

$$\overset{\ldots}{=} \frac{\lim\limits_{n \to \infty} 3 + \lim\limits_{n \to \infty} \frac{1}{n}}{\lim\limits_{n \to \infty} 7 - \lim\limits_{n \to \infty} \frac{4}{n}} = \frac{3+0}{7-0} = \frac{3}{7}.$$

The above is <u>NOT</u> a rigorous proof in the context of a Real Analysis course. The definition must be used!

However, this first (non-rigorous) step is essential because it provides us with the information that $s = \frac{3}{7}$ should be used when trying to apply the definition of convergence.

<u>Rigorous proof that the above sequence converges to $s = \frac{3}{7}$.</u>

<u>Sketch:</u> Given $\varepsilon > 0$ we must find $N \in \mathbb{N}$ such that $n \geq \underset{N = N(\varepsilon)}{N} \Longrightarrow \left| s_n - \frac{3}{7} \right| < \varepsilon$

Solve in n:

$$\left| \frac{3n+1}{7n-4} - \frac{3}{7} \right| < \varepsilon \iff \left| \frac{21n + 7 - 21n + 12}{(7n-4)\cdot 7} \right| < \varepsilon$$

$$\iff \left| \frac{19}{(7n-4)\cdot 7} \right| < \varepsilon$$

For all $n \geq 1$
$7n - 4 > 0$

$$\iff \frac{19}{(7n-4)\cdot 7} < \varepsilon \iff 19 < 7\varepsilon(7n-4)$$

$$\iff \frac{19}{7\varepsilon} < 7n - 4 \iff \frac{19}{7\varepsilon} + 4 < 7n$$

$$\iff \underbrace{\frac{19}{49\varepsilon} + \frac{4}{7}}_{N(\varepsilon)} < n$$

$$\frac{19}{49\varepsilon} + \frac{4}{7} < \underbrace{\left\lceil \frac{19}{49\varepsilon} + \frac{4}{7} \right\rceil + 1}_{\in \mathbb{N}} \leq n$$

"Official" proof: Given $\varepsilon>0$, let $N\in\mathbb{N}$ be the smallest natural number which is $> \frac{19}{49\varepsilon}+\frac{4}{7}$; that is $N=\left\lceil \frac{19}{49\varepsilon}+\frac{4}{7}\right\rceil+1$,

If $n\geq N$, then

$$n>\frac{19}{49\varepsilon}+\frac{4}{7} \Rightarrow \left|\frac{19}{(7n-4)\cdot 7}\right|<\varepsilon$$

$$\Leftrightarrow \left|s_n-\frac{3}{7}\right|<\varepsilon.$$

This shows that the definition of convergence holds with $s=\frac{3}{7}$, that is, $\lim_{n\to\infty} s_n=\frac{3}{7}$.

$\square$

Example: Prove that $\lim_{n\to\infty}\frac{1}{n^2}=0$.

$(s_n)_{n\in\mathbb{N}}$, $s_n=\frac{1}{n^2}$.

Sketch: Given $\varepsilon>0$, we need to find $N=N(\varepsilon)$ such that $n\geq N \Rightarrow \left|\frac{1}{n^2}-0\right|<\varepsilon.$ ← solve this in n.

$\varepsilon>0,\ n>0$ $\qquad$ $\sqrt{\cdot}$ incr. fct.

$$\left|\frac{1}{n^2}-0\right|<\varepsilon \Leftrightarrow \left|\frac{1}{n^2}\right|<\varepsilon \underset{n^2>0}{\Leftrightarrow} \frac{1}{n^2}<\varepsilon \Leftrightarrow \frac{1}{\varepsilon}<n^2 \Leftrightarrow \frac{1}{\sqrt{\varepsilon}}<n$$

Essentially this will be our $N=N(\varepsilon)$

"Official" proof: Given $\varepsilon>0$, let $N=\left\lceil\frac{1}{\sqrt{\varepsilon}}\right\rceil+1$, that is, $N\in\mathbb{N}$ is the smallest natural number which is $>\frac{1}{\sqrt{\varepsilon}}=\varepsilon^{-1/2}$.

If $n \geq N$, then $n > \frac{1}{\sqrt{\varepsilon}}$, so $\frac{1}{n^2} < \varepsilon$ and hence $\left|\frac{1}{n^2} - 0\right| < \varepsilon$. Therefore, the definition is satisfied with $s = 0$, that is, $\lim_{n\to\infty} \frac{1}{n^2} = 0$.

<u>Proposition:</u> If (s_n) is a sequence, such that $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} s_n = t$, then $s = t$.

"The limit of a sequence, if it exists, is <u>unique</u>"

<u>Proof:</u> Using the definition for each of the limits:

Given $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that

$$n \geq N_1 \Longrightarrow |s_n - s| < \varepsilon$$

there exists $N_2 \in \mathbb{N}$ such that

$$n \geq N_2 \Longrightarrow |s_n - t| < \varepsilon$$

Take $N = \max\{N_1, N_2\} \in \mathbb{N}$. If $n \geq N$, then both of the above hold, so: $|s_n - s| < \varepsilon$ and $|s_n - t| < \varepsilon$.

$$|s - t| = |\underbrace{(s - s_n)}_{= a} + \underbrace{(s_n - t)}_{= b}| \leq |\underbrace{s - s_n}_{= a}| + |\underbrace{s_n - t}_{= b}|$$

Triangle inequality $|a + b| \leq |a| + |b|$

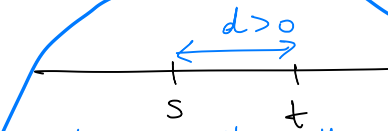
$$= \underbrace{|s_n - s|}_{< \varepsilon} + \underbrace{|s_n - t|}_{< \varepsilon}$$

$$< \varepsilon + \varepsilon = 2\varepsilon.$$

We conclude that for all $\varepsilon > 0$,

$|s-t| < 2\varepsilon$. Therefore $|s-t| = 0$

i.e., $s = t$.



If $d = |s-t| > 0$, then take $\varepsilon = d/4 > 0$.
$|s-t| = d = 4\varepsilon$, which is not $< 2\varepsilon$

$\square$

Example:

Consider the sequence $(a_n)_{n \geq 0}$, $a_n = (-1)^n$.

Claim: This sequence does not admit a limit. (i.e., it does not converge).

Proof: (by Contradiction). Suppose that there exists $a \in \mathbb{R}$ such that $\lim_{n \to \infty} a_n = a$. Then for $\varepsilon = 1$, there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow |a_n - a| < 1$.

i.e. $n \geq N \Rightarrow |(-1)^n - a| < 1$.

Recall $a_n = (-1)^n = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$, so the above implies that, if $n \geq N$, then

$$\underbrace{|\overset{=a_n}{1} - a| \overset{(*)}{<} 1}_{\text{Take } n \geq N \text{ even}} \quad \text{and} \quad \underbrace{|\overset{=a_n}{-1} - a| \overset{(**)}{<} 1}_{\text{Take } n \geq N \text{ odd}}$$

But then, we find that:

$$2 = |1-(-1)| = |\underbrace{(1-a)} + \underbrace{(a-(-1))}| \leq$$

Triangle inequality

$$\leq \underbrace{|\underbrace{1-a}|}_{\substack{<1 \\ \text{by } (*)}} + \underbrace{|\underbrace{a-(-1)}_{a+1}|}_{\substack{<1 \\ \text{by } (**)}} < 2$$

This contradiction shows that the assumption that there exists $a \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n = a$ is <u>false</u>; therefore the sequence (a_n) does not converge. $\square$