MAT320/640 Lecture 6 9/22/2021

More sequences

Recall that a sequence $(s_n)_{n\geq 1}$ **converges** to $s\in\mathbb{R}$ if for all $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that if $n\geq N$, then $|s_n-s|<\varepsilon$.

could be $n\geq m$

i.e., the set $\{s_n\in\mathbb{R} : n\geq 1\}$ is bounded.

Proposition. Convergent sequences are bounded.

Proof: Since $(s_n)_{n\geq1}$ is convergent, letting $\varepsilon=1$, there exists $N\in\mathbb{N}$ such that if $n\geq N$, then $|s_n-s|<1$. By the triangle inequality,

$$|s_n|-|s|\leq|s_n-s|<1$$

$|a+b|\leq|a|+|b|$

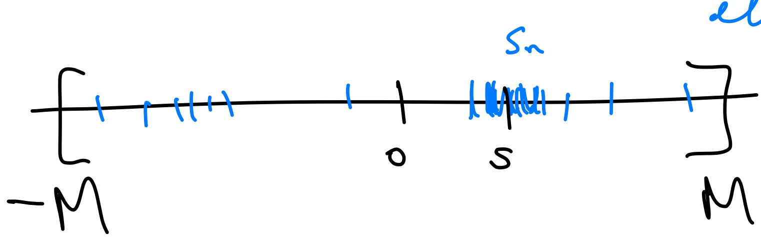
so: $|a+b|-|b|\leq|a|$

Therefore $|s_n|\leq 1+|s|$, for all $n\geq N$.

$a+b=s_n$ $\quad$ $a=s_n-s$

$b=s$

Let
$$M=\max\{\underbrace{|s_1|,|s_2|,\dots,|s_N|}_{\text{Takes care of first few elements } (n\leq N)},\ \underbrace{1+|s|}_{\text{takes care of all elements with } n\geq N}\}.$$



From the above, it follows that for all $n\in\mathbb{N}$, $|s_n|\leq M$; i.e. $\{s_n : n\in\mathbb{N}\}$ is bounded. $\square$

Operations with sequences

Start here

Proposition. If $(s_n)_{n\geq 1}$ converges to s, and $k \in \mathbb{R}$, then $(k\cdot s_n)_{n\geq 1}$ converges to $k\cdot s$.

End here

Proof: We need to show for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, if $n \geq N$ then $|k\cdot s_n - k\cdot s| < \varepsilon$.
We know that for all $\tilde{\varepsilon} > 0$, there exists $\tilde{N} \in \mathbb{N}$ such that if $n \geq \tilde{N}$ then $|s_n - s| < \tilde{\varepsilon}$.
If $k = 0$, then $k\cdot s_n = 0$ for all s_n, and $k\cdot s = 0$, so there is nothing to prove. So, suppose $k \neq 0$.
Then, given $\varepsilon > 0$, let $\tilde{\varepsilon} = \frac{\varepsilon}{k}$; by what we know, there exists $\tilde{N} \in \mathbb{N}$ such that if $n \geq \tilde{N}$, then $|s_n - s| < \tilde{\varepsilon} = \frac{\varepsilon}{k}$; thus $k \cdot |s_n - s| < \varepsilon$, i.e. $|k\cdot s_n - k\cdot s| < \varepsilon$. Thus, we can take $N = \tilde{N}$ and we find that $|k\cdot s_n - k\cdot s| < \varepsilon$, for the given $\varepsilon > 0$, as desired. $\square$

Start here (know)

Proposition. If $(s_n)_{n\geq 1}$ converges to s, and $(t_n)_{n\geq 1}$ converges to t, then $(s_n + t_n)_{n\geq 1}$ converges to $s + t$.

End here. (need)

Proof: We need to show that given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$, then

$$\left|(s_n + t_n) - (s+t)\right| < \varepsilon$$

We know that given $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, there exist $N_1, N_2 \in \mathbb{N}$ such that

$$n \geq N_1 \implies |s_n - s| < \varepsilon_1$$

$$n \geq N_2 \implies |t_n - t| < \varepsilon_2$$

Let $\varepsilon > 0$ be given, take $\varepsilon_1 = \frac{\varepsilon}{2}$ and $\varepsilon_2 = \frac{\varepsilon}{2}$. By what we know, there exist $N_1, N_2 \in \mathbb{N}$ such that $n \geq \max\{N_1, N_2\} =: N$, then

$$|s_n - s| < \frac{\varepsilon}{2} \quad \text{and} \quad |t_n - t| < \frac{\varepsilon}{2}; \quad \text{so}$$

$$\left|(s_n + t_n) - (s+t)\right| \leq |s_n - s| + |t_n - t|$$

Triangle ineq.

$$|a+b| < |a| + |b|$$

$$a = s_n - s$$
$$b = t_n - t$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

as desired. $\square$

More operations with sequences

Proposition. If $\boxed{(s_n)_{n\geq 1} \text{ converges to } s, \text{ and } (t_n)_{n\geq 1} \text{ converges to } t,}$ ← know / start
then $\boxed{(s_n \cdot t_n)_{n\geq 1} \text{ converges to } s\cdot t.}$ ← need/end

Proposition. If $\boxed{(s_n)_{n\geq 1} \text{ converges to } s\neq 0, \text{ and } s_n \neq 0 \text{ for all } n\geq 1,}$ ← start/know
then $\boxed{\left(\frac{1}{s_n}\right)_{n\geq 1} \text{ converges to } \frac{1}{s}.}$ ← need/end

Proposition. If $\boxed{(s_n)_{n\geq 1} \text{ converges to } s\neq 0 \text{ and } s_n\neq 0 \text{ for all } n\geq 1, \text{ and } (t_n)_{n\geq 1} \text{ converges to } t,}$ ← start/know
then $\boxed{\left(\frac{t_n}{s_n}\right)_{n\geq 1} \text{ converges to } \frac{t}{s}.}$ ← need/end

Using the above, we can study the following:

Example 1. $\lim\limits_{n\to\infty} \frac{1}{n^p} = 0$, for all $p>0$.

Given $\varepsilon>0$, let $N = \left\lceil \left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}} \right\rceil + 1$, i.e., $N\in\mathbb{N}$ is the smallest natural number $\geq \left(\frac{1}{\varepsilon}\right)^{1/p}+1$.
If $n\geq N > \left(\frac{1}{\varepsilon}\right)^{1/p}$, then

$$\left|\frac{1}{n^p} - 0\right| = \frac{1}{n^p} < \varepsilon.$$

□

Example 2. $\lim_{n\to\infty} a^n = 0$, if $|a|<1$

If $a=0$, there is nothing to prove, since then $a^n = 0$ for all n; so suppose $a \neq 0$.

Then $0<|a| = \frac{1}{1+b}$ for some $b>0$. Recall

$$(1+b)^n \overset{\text{Binomial thm}}{=} \sum_{k=0}^{n} \binom{n}{k} b^k \cdot \underbrace{1^{n-k}}_{=1} = 1 + \underbrace{\binom{n}{1}}_{=n} \cdot b + \cdots + b^n$$

$$> 1 + n\cdot b > n\cdot b$$

Thus
$$|a^n - 0| = |a|^n = \frac{1}{(1+b)^n} < \frac{1}{n\cdot b}$$

Given $\varepsilon>0$, let $N = \left\lceil \frac{1}{b\cdot\varepsilon} \right\rceil + 1$. Then if $n \geq N > \frac{1}{b\cdot \varepsilon}$, it follows that

$$|a^n - 0| < \frac{1}{n\cdot b} < \varepsilon.$$

Similarly, one can show:

$$\lim_{n\to\infty} (n)^{\frac{1}{n}} = 1 \quad \text{and} \quad \lim_{n\to\infty} a^{\frac{1}{n}} = 1,$$

for all $a>0$.

Using the results (Propositions and Examples) above, obtained by verifying the definition of convergence, we may now compute certain limits rigorously even if we do not verify the definition directly in these cases:

$$\lim_{n\to\infty}\frac{(3n^2+5n+1)\cdot\frac{1}{n^2}}{(7n^2+8n+3)\cdot\frac{1}{n^2}} = \lim_{n\to\infty}\frac{3+\frac{5}{n}+\frac{1}{n^2}}{7+\frac{8}{n}+\frac{3}{n^2}}$$

$$= \frac{\lim\limits_{n\to\infty} 3+\frac{5}{n}+\frac{1}{n^2}}{\lim\limits_{n\to\infty} 7+\frac{8}{n}+\frac{3}{n^2}} = \frac{3+\lim\limits_{n\to\infty}\frac{5}{n}+\lim\limits_{n\to\infty}\frac{1}{n^2}}{7+\lim\limits_{n\to\infty}\frac{8}{n}+\lim\limits_{n\to\infty}\frac{3}{n^2}}$$

$$= \frac{3+5\underbrace{\lim\limits_{n\to\infty}\frac{1}{n}}_{=0}+\underbrace{\lim\limits_{n\to\infty}\frac{1}{n^2}}_{=0}}{7+8\underbrace{\lim\limits_{n\to\infty}\frac{1}{n}}_{=0}+3\underbrace{\lim\limits_{n\to\infty}\frac{1}{n^2}}_{=0}} = \frac{3}{7}.$$

In HW problems, it will be specified if you need to use the definition or not.

$$\lim_{n\to\infty}\left(\frac{\sqrt{2}}{3}\right)^n\cdot(5n)^{\frac{1}{n}} = \underbrace{\left(\lim_{n\to\infty}\left(\frac{\sqrt{2}}{3}\right)^n\right)}_{=0}\cdot\underbrace{\left(\lim_{n\to\infty}5^{\frac{1}{n}}\right)}_{=1}\underbrace{\left(\lim_{n\to\infty}n^{\frac{1}{n}}\right)}_{=1}$$

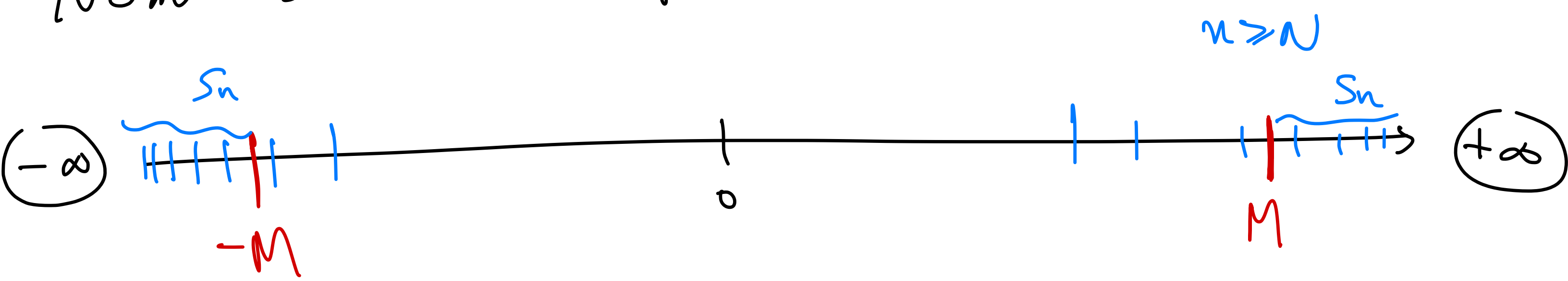
$a=\frac{\sqrt{2}}{3}$

$|a|<1$

$$=0.$$

Infinite limits

Definition. A sequence $(s_n)_{n\geq 1}$ diverges to $+\infty$ if for all $M>0$, there exists $N\in\mathbb{N}$ such that if $n\geq N$ then $s_n > M$. Similarly, $(s_n)_{n\geq 1}$ diverges to $-\infty$ if for all $M>0$, there exists $N\in\mathbb{N}$ such that if $n\geq N$, then $s_n<-M$.



Write: $\lim_{n\to\infty} s_n = \pm\infty$ (accordingly).

Example. Show using above definition that

$$\lim_{n\to\infty} \sqrt{n}+7 = +\infty.$$

Given $M>0$, we need to find $N\in\mathbb{N}$ such that $n\geq N$ implies $\sqrt{n}+7>M$. Take $N\in\mathbb{N}$ to be $N=\lceil (M-7)^2\rceil+1$, i.e., N is the smallest integer larger than $(M-7)^2+1$. If $n\geq N$, then $n>(M-7)^2$, so $\sqrt{n}>M-7$ i.e. $\sqrt{n}+7>M$. $\square$

know

Proposition. If $(s_n)_{n\geq 1}$ and $(t_n)_{n \geq 1}$ are sequences such that $\lim_{n\to\infty} s_n = +\infty$ and $\lim_{n\to\infty} t_n > 0$, then

$$\boxed{\lim_{n\to\infty} s_n \cdot t_n = +\infty.}$$ ← need

Proof. We need to show that given $M>0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $s_n \cdot t_n > M$.

Since $\lim_{n\to\infty} t_n > 0$, choose $m>0$ such that $0 < m < \lim_{n\to\infty} t_n$. Then $t_n > m$ for sufficiently large n, that is, there exists $N_1 \in \mathbb{N}$ such that $M \geq N_1$, then $t_n > m$. Since $\lim_{n\to\infty} s_n = +\infty$, given $\tilde{M} = \frac{M}{m}$, there exists $N_2 \in \mathbb{N}$ such that $n \geq N_2$, $s_n > \tilde{M} = \frac{M}{m}$, i.e. $m \cdot s_n > M$.

Take $N = \max\{N_1, N_2\}$, then $n \geq N$ so

$$s_n \cdot t_n \underset{t_n > m}{>} s_n \cdot m > M.$$

$\square$

$s_n > 0$ for all $n \geq 1$

Proposition. A sequence $(s_n)_{n\geq 1}$ of positive numbers satisfies $\lim_{n\to\infty} s_n = +\infty$ if and only if $\lim_{n\to\infty} \frac{1}{s_n} = 0$.