MAT 320/640 # Lecture 8 9/29/2021

Recap from last lecture:

Def: $(s_n)_{n\in\mathbb{N}}$ is Cauchy if $\forall \varepsilon>0\ \exists N\in\mathbb{N}$ such that if $n,m\geq N$ then $|s_n-s_m|<\varepsilon$.

We proved last time that:

$$(s_n)_{n\in\mathbb{N}} \text{ is convergent} \Longrightarrow (s_n)_{n\in\mathbb{N}} \text{ is Cauchy}$$

Prop. If $(s_n)_{n\in\mathbb{N}}$ is Cauchy, then $(s_n)_{n\in\mathbb{N}}$ is bounded.

Pf: Take $\varepsilon=1$; then there exists $N\in\mathbb{N}$ s.t. if $n,m\geq N$, then $|s_n-s_m|<\varepsilon=1$, i.e.

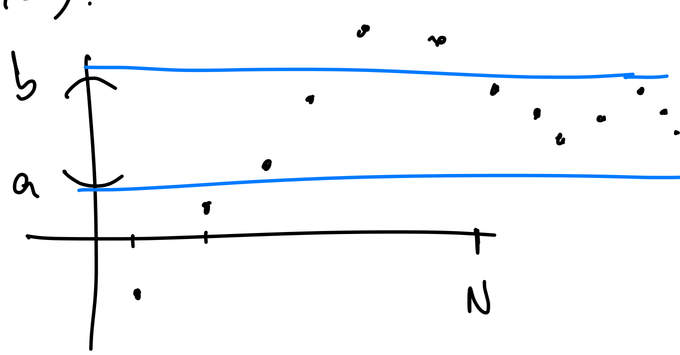
$$-1<s_n-s_m<1$$

Take $m=N$: $\quad -1<s_n-s_N<1$

$$\underbrace{s_N-1}_{=a}<s_n<\underbrace{s_N+1}_{=b} \quad \text{for all } n\geq N$$

So, for $n\geq N$, all elements in the sequence are in the interval $(s_N-1, s_N+1)=(a,b)$.

Let $M=\max\limits_{1\leq n\leq N}\{|s_n|\}$. Then



$|s_n|\leq M$ if $n\leq N$

$a<s_n<b$ if $n\geq N$

i.e., $(s_n)_{n\in\mathbb{N}}$ is bounded. $\square$

<u>Theorem</u>. $(s_n)_{n\in\mathbb{N}}$ is Cauchy $\iff$ $(s_n)_{n\in\mathbb{N}}$ is convergent.

(This is equivalent to the <u>Completeness Axiom</u>)

existence of sup of any nonempty bounded subset of $\mathbb{R}$.

<u>Proof</u>. First, recall we already proved that convergent sequences are Cauchy. $(\Longleftarrow)$. For the converse $(\Longrightarrow)$, we will use that if $(s_n)_{n\in\mathbb{N}}$ is a sequence such that $\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = L$, then the sequence is convergent and $\lim_{n\to\infty} s_n = L$.

Since $(s_n)_{n\in\mathbb{N}}$ is Cauchy, for any given $\varepsilon>0$, $\exists N\in\mathbb{N}$ s.t. if $n,m\geq N$, then

$|s_n - s_m| < \varepsilon$, so $-\varepsilon < s_n - s_m < \varepsilon$ and hence

$s_n < s_m + \varepsilon$. That implies that:

$$v_N := \sup\{s_n : n\geq N\} \leq s_m+\varepsilon \quad \text{for } m\geq N.$$

Thus $\quad v_N - \varepsilon \leq s_m$ for all $m\geq N$, hence

$$v_N - \varepsilon \leq \inf\{s_m : m\geq N\}. \text{ Altogether:}$$

$$\underline{\sup\{s_n : n\geq N\}} = v_N \leq \underline{\inf\{s_m : m\geq N\}} + \varepsilon$$

Taking the limit as $N\to\infty$ of both sides,

since
$$\limsup_{n\to\infty} s_n = \lim_{N\to\infty} \sup\{s_n : n \geq N\}$$
$$\liminf_{n\to\infty} s_n = \lim_{N\to\infty} \inf\{s_m : m \geq N\}$$

We get:
$$\limsup_{n\to\infty} s_n \leq \left(\liminf_{n\to\infty} s_n\right) + \varepsilon$$

We may choose this arbitrarily small.

and thus $\limsup_{n\to\infty} s_n \leq \liminf_{n\to\infty} s_n$. Thus the sequence $(s_n)_{n\in\mathbb{N}}$ converges, and
$$\lim_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n .$$
$\square$

Intuition: $s_1, s_2, s_3, s_4, s_5, s_6, s_7, \ldots$

$s_1, s_2, s_4, s_7, \ldots$

Subsequences

Def: If $(s_n)_{n\in\mathbb{N}}$ is a sequence, then a subsequence of $(s_n)_{n\in\mathbb{N}}$ is a sequence $(s_{n_k})_{k\in\mathbb{N}}$, where the assignment $k \longmapsto n_k$ is an increasing function $\mathbb{N}\to\mathbb{N}$.

$s_1, s_2, s_3, s_4, s_5, s_6, s_7, \ldots$

$s_1, s_2, s_4, s_7, \ldots$

$n_1 = 1,\ n_2 = 2,\ n_3 = 4,\ n_4 = 7, \ldots$

n	1	2	3	4	5	6	7	…
k	1	2	3	4	…			
n_k	1	2	4	7	…			

Example: $s_n = (-1)^n \cdot n^2$

n	s_n
1	-1
2	4
3	-9
4	16
5	-25
$\vdots$	$\vdots$

The seq. s_n alternates sign, so there are infinitely many positive and infinitely many negative elements.

For example, we may consider the subsequence of <u>all positive elements</u> of s_n

k	n_k	s_{n_k}
1	2	4
2	4	16
3	6	36
4	8	64
5	10	100
$\vdots$	$\vdots$	$\vdots$
k	$2k$	$s_{2k} = (-1)^{2k} \cdot (2k)^2$

i.e. $n_k = 2k$.

$$\boxed{s_{n_k} = 4k^2}$$

Another subsequence of (s_n) is the subsequence of all <u>negative elements</u>

$$s_{n_\ell} = (-1)^{2\ell-1} \cdot (2\ell-1)^2 = -(2\ell-1)^2$$

$n_\ell = 2\ell - 1$

ℓ	n_ℓ	s_{n_ℓ}
1	1	-1
2	3	-9
3	5	-25
4	7	-49
$\vdots$	$\vdots$	$\vdots$

Example: $a_n = \sin\left(\frac{n\pi}{4}\right) \qquad n \in \mathbb{N}$

n	a_n
1	$\sqrt{2}/2$
2	1
3	$\sqrt{2}/2$
4	0
5	$-\sqrt{2}/2$
6	-1
7	$-\sqrt{2}/2$
8	0
$\vdots$	$\vdots$

this sequence is periodic:

$a_{n+8} = a_n$
$\forall n \in \mathbb{N}$.

Examples of subsequences of a_n:

- subsequence of 1's only

$\{1, 1, 1, 1, \ldots\}$

$\{a_2, a_{10}, a_{18}, \ldots\}$

$(a_{n_k})_{k\in\mathbb{N}}$, where $n_k = 8(k-1)+2$
i.e. $n_k = 8k-6$.

- subsequence of odd indices:

$(a_{n_\ell})_{\ell\in\mathbb{N}}$, $\quad n_\ell = 2\ell - 1$.

$\{a_1, a_3, a_5, a_7, \ldots\} =$

$= \left\{\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \ldots\right.$

- subsequence of even indices

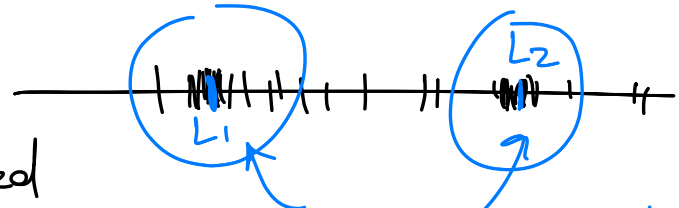
$(a_{n_t})_{t\in\mathbb{N}} \qquad n_t = 2t$.

$\{a_2, a_4, a_6, a_8, \ldots\} =$

$= \{1, 0, -1, 0, 1, 0, -1, 0, \ldots$

Theorem. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence.

- Given $t\in\mathbb{R}$, there exists a subsequence $(s_{n_k})_{k\in\mathbb{N}}$ of $(s_n)_{n\in\mathbb{N}}$ that converges to t if and only if $\{n\in\mathbb{N} : |s_n - t| < \varepsilon\}$ is an infinite set for all $\varepsilon > 0$.



$(s_n)_{n\in\mathbb{N}}$ enters these arbitrarily small neighbd's of L_1 and L_2 infinitely often if and only if there exist subsequences converging to L_1 and to L_2.

- If $(s_n)_{n\in\mathbb{N}}$ is unbounded from above, then it has a subsequence that diverges to $+\infty$.

- If $(s_n)_{n\in\mathbb{N}}$ is unbounded from below, then it has a subsequence that diverges to $-\infty$

Theorem. If a sequence $(s_n)_{n\in\mathbb{N}}$ converges to L, then every subsequence of $(s_n)_{n\in\mathbb{N}}$ also converges to L.

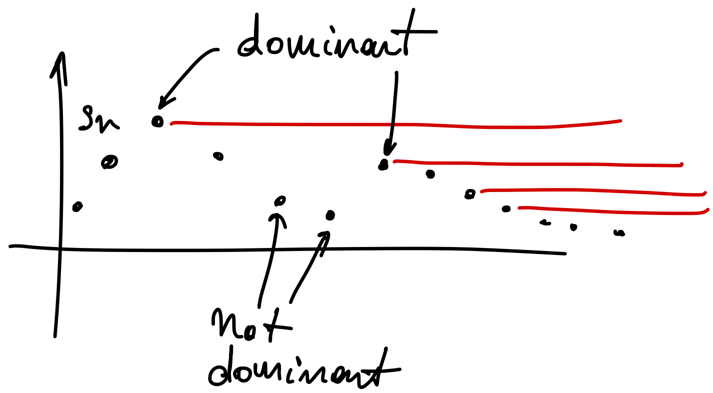
Pf. Let $(s_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(s_n)_{n\in\mathbb{N}}$. Since $s_n \longrightarrow L$, we know that $\forall \varepsilon>0$, $\exists N\in\mathbb{N}$ s.t. if $n\geq N$, then $|s_n - L| < \varepsilon$. As $k\longmapsto n_k$ is an increasing function, $n_k \geq k$ for all $k\in\mathbb{N}$. Thus, $n_k \geq k \geq N$ whenever $k\geq N$. Therefore $|s_{n_k} - L| < \varepsilon$ whenever $k\geq N$, i.e., $s_{n_k} \longrightarrow L$. $\square$

Remark: The converse statement also holds!

> **Bolzano–Weierstrass Theorem**
> Every bounded sequence (in $\mathbb{R}$) has a convergent subsequence.

Lemma. Every sequence $(s_n)_{n\in\mathbb{N}}$ has a monotonic subsequence.

Pf. Let's say s_n is a dominant term if $\forall m>n$, $s_n > s_m$. Clearly, there are 2 possibilities:



1) There are infinitely many dominant elements in $(s_n)_{n\in\mathbb{N}}$
2) There are finitely many dominant elements in $(s_n)_{n\in\mathbb{N}}$

In case 1), let $(s_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(s_n)_{n\in\mathbb{N}}$ consisting only of dominant elements.

By definition, $s_{n_k} > s_{n_{k+1}}$ for all k; so $(s_{n_k})_{k\in\mathbb{N}}$ is monotonic decreasing

In case 2), since there are only finitely many dominant elements, we may choose $n_1 \in \mathbb{N}$ such that s_n is <u>not</u> dominant for all $n \geq n_1$.

no dominant element after s_{n_1} $\iff$ $\forall N \geq n_1,\ \exists m \in \mathbb{N}$ $s_m \geq s_N$

i.e. s_N is <u>not</u> dominant.

Apply the same reasoning with $N = n_1$: choose $n_2 \in \mathbb{N}$ to be the $m \in \mathbb{N}$ given above; i.e.

$$s_{n_2} \geq s_{n_1}$$

and proceed inductively. This produces a monotonic increasing subsequence $s_{n_1} \leq s_{n_2} \leq s_{n_3} \leq \dots$

$\square$

<u>Proof of Bolzano–Weierstrass Thm:</u>

Since $(s_n)_{n \in \mathbb{N}}$ is bounded, so is any subsequence $(s_{n_k})_{k \in \mathbb{N}}$. Choose a monotonic subsequence $(s_{n_k})_{k \in \mathbb{N}}$, which exists by the above Lemma. Then $(s_{n_k})_{k \in \mathbb{N}}$ is bounded and monotonic, and <u>therefore</u>, convergent. $\square$

We proved in Lecture 7 that monotonic bounded sequences converge.

Subsequential limits

Definition. A subsequential limit of a sequence $(s_n)_{n\in\mathbb{N}}$ is a real number or $\pm\infty$ which arises as the limit of some subsequence of $(s_n)_{n\in\mathbb{N}}$.

We denote $S = \{t \in \mathbb{R} \cup \{\pm\infty\} : s_{n_k} \to t$ for some subsequence $(s_{n_k})_{k\in\mathbb{N}}$ of $(s_n)_{n\in\mathbb{N}}\}$

Examples:

- $s_n = (-1)^n \cdot n^2$

n	s_n
1	-1
2	4
3	-9
4	16
5	-25
$\vdots$	$\vdots$

All subsequences of $(s_n)_{n\in\mathbb{N}}$ that have a limit, go either to $+\infty$ or $-\infty$.

$$S = \{-\infty, +\infty\}.$$

- $a_n = \sin\left(\frac{n\pi}{4}\right)$

n	a_n
1	$\sqrt{2}/2$
2	1
3	$\sqrt{2}/2$
4	0
5	$-\sqrt{2}/2$
6	-1
7	$-\sqrt{2}/2$
8	0
$\vdots$	$\vdots$

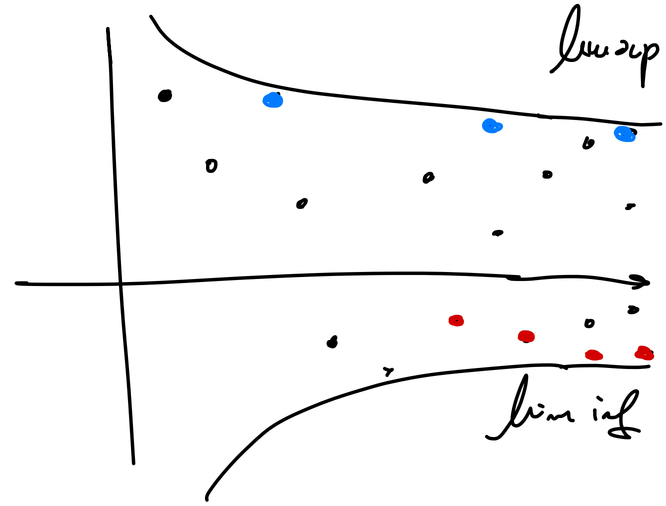
All subsequences of $(a_n)_{n\in\mathbb{N}}$ that have a limit are eventually constant (because $(a_n)_{n\in\mathbb{N}}$ is periodic).

$$S = \left\{-1, -\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 1\right\}.$$

Theorem. If $(s_n)_{n\in\mathbb{N}}$ is a sequence then there exist monotonic subsequences $(s_{n_k})_{k\in\mathbb{N}}$ and $(s_{n_\ell})_{\ell\in\mathbb{N}}$ which converge to $\limsup_{n\to\infty} s_n$ and $\liminf_{n\to\infty} s_n$ respectively:

$$\lim_{k\to\infty} \underline{s_{n_k}} = \limsup_{n\to\infty} s_n$$

$$\lim_{\ell\to\infty} \underline{s_{n_\ell}} = \liminf_{n\to\infty} s_n$$



Theorem. If $(s_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}$, denoting by S the set of its subsequential limits, $S \neq \emptyset$ and

$$\limsup_{n\to\infty} s_n = \sup S$$

$$\liminf_{n\to\infty} s_n = \inf S$$

In particular, $s_n \to L$ if and only if $S = \{L\}$.

[Note: this last statement is the converse mentioned in the Remark above).