MAT 330/681 Lecture 11 3/10/2021

"Types" of random variables ("distributions")

$\to$ discrete: Bernoulli, Binomial, Poisson, geometric, ...

$\to$ continuous (later)

Bernoulli Random Variable

A discrete random variable X is Bernoulli with parameter p

write: $X \sim \text{Bernoulli}(p)$

if:

- $X : \Omega \longrightarrow \{0, 1\}$.

  (X can assume only values 0 and 1)

  "failure" "off" / "success" "on"

- $P(X = 0) = 1 - p$,

  $P(X = 1) = p$.

- Expected Value: $E(X) = \sum_x x \cdot P(X = x) = 0 \cdot \underbrace{P(X=0)}_{1-p} + 1 \cdot \underbrace{P(X=1)}_{p}$

  $= \boxed{p.}$

- Variance: $Var(X) = E(X^2) - E(X)^2$

  $$= \left(\sum_x x^2 P(X = x) \right) - p^2$$

  $$= 0^2 \cdot \underbrace{P(X=0)}_{1-p} + 1^2 \cdot \underbrace{P(X=1)}_{p} - p^2$$

  $$= p - p^2 = \boxed{p(1-p).}$$

Binomial random variable

A discrete random variable X is Binomial with parameters n and p if it is a sum of n independent Bernoulli random variables with parameter p.

$$X_1, X_2, X_3, \dots, X_n \sim \text{Bernoulli}(p)$$

$$X = \underset{\substack{\uparrow \\ 1^{st}\text{ trial}}}{X_1} + \underset{\substack{\uparrow \\ 2^{nd}\text{ trial}}}{X_2} + X_3 + \dots + \underset{\substack{\uparrow \\ n^{th}\text{ trial}}}{X_n} \sim \text{Binomial}(\underset{\#\text{ of trials}}{n}, \underset{\text{prob. of success.}}{p})$$

Recalling our computations on Bernoulli process:

$$P(X = i) = \binom{n}{i} \cdot p^i (1-p)^{n-i} \longleftarrow \text{prob. mass function}$$

$\left(\begin{matrix} i \in \mathbb{Z} \\ 0 \le i \le n \end{matrix}\right)$

- Expected value: $E(X) = E(X_1 + \dots + X_n)$

E(·) is linear

$$\overset{\text{E(·) is linear}}{=} \underbrace{E(X_1)}_{p} + \dots + \underbrace{E(X_n)}_{p} = \boxed{n \cdot p.}$$

$X_i \sim \text{Bernoulli}(p)$

- Variance: $Var(X) = Var(X_1 + \dots + X_n)$

$$= \underbrace{Var(X_1)}_{p(1-p)} + \dots + \underbrace{Var(X_n)}_{p(1-p)} + \underbrace{Cov(X_1, X_2) + \dots}_{=0}$$

$Cov(X_i, X_j) = 0$ if $i \neq j$ b/c X_i are independent

$$= \boxed{n \cdot p(1-p).}$$

Ex: If 10% of products manufactured at a certain industry are defective, what is the probability that a random batch of 10 products from this industry have

a) no defective products?

b) at least 3 defective products?

Let X be the random variable that counts the # of defective products in a batch of 10 products.

$$X = X_1 + \cdots + X_{10} \sim \text{Binomial}\left(10, \frac{1}{10}\right)$$

$$X_i \sim \text{Bernoulli}\left(\frac{1}{10}\right), \quad i = 1, \ldots, 10$$

a) $P(X=0) = \binom{10}{0} p^0 (1-p)^{10} = \left(\frac{9}{10}\right)^{10} \cong 0.3487 = \boxed{34.87\%}$

b) $$\begin{aligned} P(X \geq 3) &= 1 - P(X \leq 2) \\ &= 1 - P(X=0) - P(X=1) - P(X=2) \\ &= 1 - \binom{10}{0} p^0 (1-p)^{10} - \binom{10}{1} p^1 (1-p)^9 - \binom{10}{2} p^2 (1-p)^8 \\ &= 1 - \left(\frac{9}{10}\right)^{10} - 10 \frac{1}{10} \left(\frac{9}{10}\right)^9 - 45 \frac{1}{10^2} \left(\frac{9}{10}\right)^8 \\ &\cong 0.0702 = \boxed{7.02\%} \end{aligned}$$

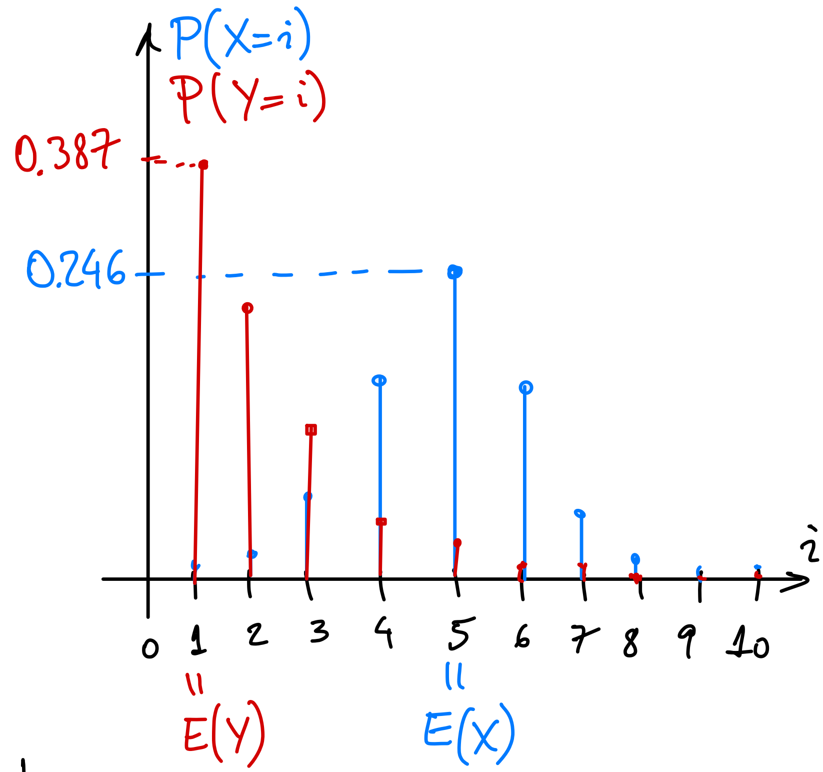
Comparing Binomial Random Variables with different parameters

$X \sim \text{Binomial}\left(10, \frac{1}{2}\right)$

$Y \sim \text{Binomial}\left(10, \frac{1}{10}\right)$

$E(X) = n \cdot p = 10 \cdot \frac{1}{2} = 5$

$E(Y) = n \cdot p = 10 \cdot \frac{1}{10} = 1$



Use the link below to play with Binomial distr. w/ different parameters n, p on your own!

Ex. A board of directors has 12 members, and at least 8 must agree to a business proposal for it to go forward. All directors act independently and make the correct decision with probability p. Suppose a business proposal has probability α of being good for the company. What is the prob. that the board of directors agrees to it?

α = prob. of decision being good $\qquad 1-\alpha$ = prob. of decision being bad.

Consider the events $\begin{cases} B = \text{Board agrees to the decision.} \\ G = \text{Decision is good} \end{cases}$

$$P(B) = P(B|G)\underbrace{P(G)}_{\alpha} + P(B|G^c)\underbrace{P(G^c)}_{1-\alpha}$$

- If decision is good:

$$P(B|G) = \sum_{i=8}^{12} \binom{12}{i} p^i (1-p)^{12-i}$$

Prob. of a Binomial(12, p) random variable being ≥ 8

(need at least 8 directors to make the correct decision)

- If decision is bad:

$$P(B|G^c) = \sum_{i=5}^{12} \binom{12}{i} p^i (1-p)^{12-i}$$

Prob. of a Binomial(12, p) random variable being ≥ 5

(need at least 5 directors to make the correct decision, i.e., not more than 8 to make wrong decision)

Altogether

$$P(B) = \alpha \cdot \sum_{i=8}^{12} \binom{12}{i} p^i (1-p)^{12-i} + (1-\alpha) \sum_{i=5}^{12} \binom{12}{i} p^i (1-p)^{12-i}$$

e.g., if $\alpha = 50\%$ and $p = 80\%$, then

$$P(B) = 96.34\%$$