MAT 330/681 Lecture 18 4/12/2021

"Geometric Problems"

Ex: Suppose that friends X and Y are meeting online on Zoom to hang out during quarantine, and planned to join the room sometime between 3:00pm and 4:00pm. Assume X logs in at a time unif. distr. between 3:15pm and 3:45pm, while Y logs in at a time unif. distr. between 3:00pm and 4:00pm. What is the prob. that:

a) friend that logs in first has to wait < 5 min for the other friend?

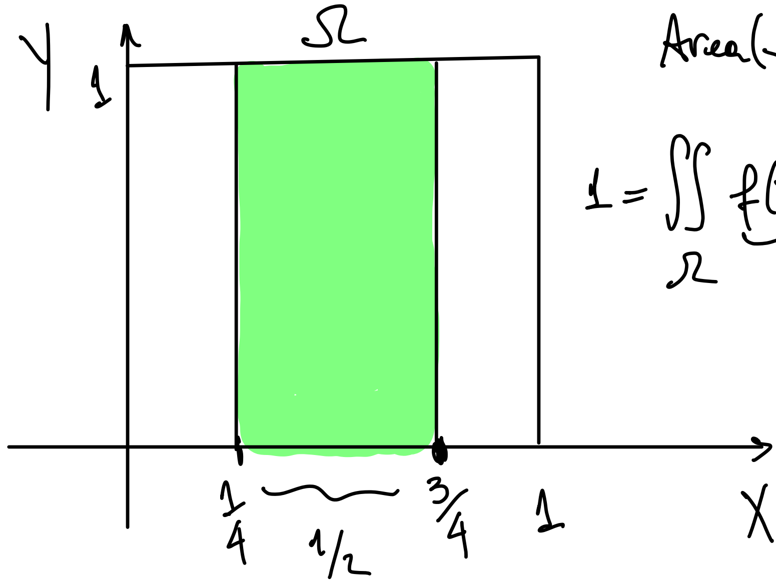
b) X arrives first?

$\frac{1}{4}$ hour = 15 min

$\frac{3}{4}$ hour = 45 min

$$X \sim \text{Uniform}\left(\left[\frac{1}{4}, \frac{3}{4}\right]\right)$$

$$Y \sim \text{Uniform}([0,1])$$

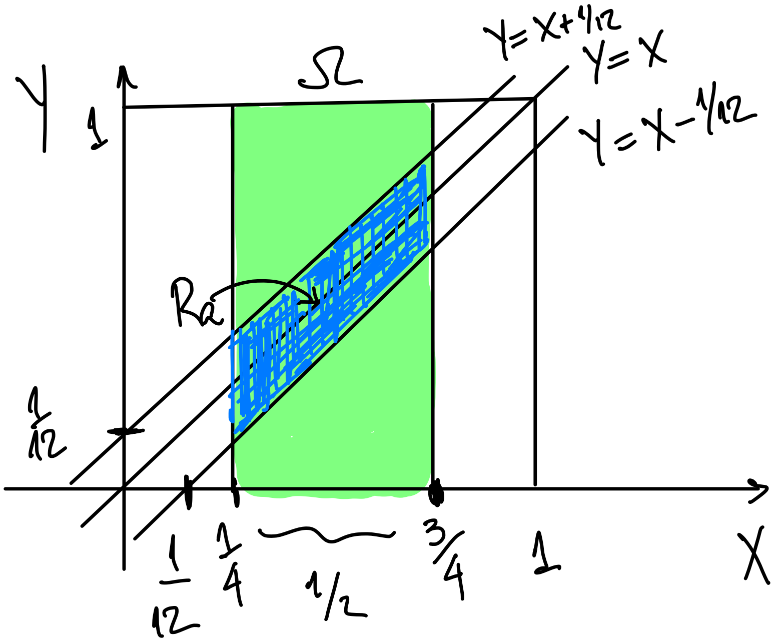


$$\text{Area}(\Omega) = \frac{1}{2} \cdot 1 = 1/2.$$

$$1 = \iint_{\Omega} \underbrace{f(x,y)}_{c}\, dx\,dy = c \cdot \text{Area}(\Omega) = \frac{c}{2}.$$

$\Rightarrow f(x,y) = c = 2$ is the joint prob. density function of X and Y.

a) $$|X - Y| = \text{wait time} < \frac{5}{60} = \frac{1}{12}$$



$$|X-Y| < \frac{1}{12} \iff$$

$$\iff -\frac{1}{12} < X - Y < \frac{1}{12}$$

$$\iff -X-\frac{1}{12} < -Y < -X+\frac{1}{12}$$

$$\iff X+\frac{1}{12} > Y > X-\frac{1}{12}$$

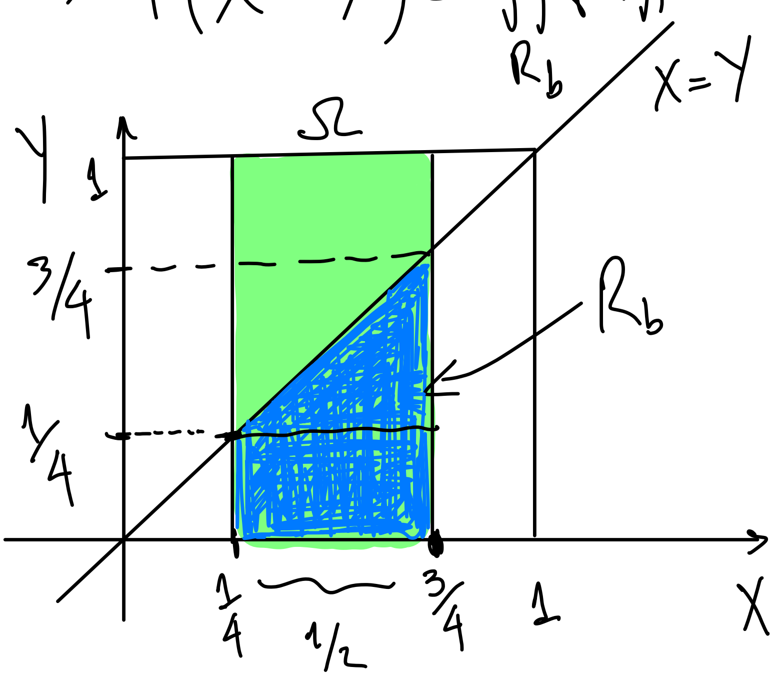
$$\iff \boxed{X-\frac{1}{12} < Y < X+\frac{1}{12}}$$

a) $$P(\text{wait time} < 5) = \iint_{R_a} f(x,y)\,dx\,dy = 2 \underbrace{\iint_{R} dx\,dy}_{\text{Area}(R_a)}$$

$$= 2 \cdot \underbrace{\left(2 \cdot \frac{1}{12}\right) \cdot \frac{1}{2}}_{\text{Area}(R_a)} = \boxed{\frac{1}{6}} = \underline{16.67\%}$$

b) $$P(X<Y) = \iint_{R_b} f(x,y)\,dx\,dy = 2\,\text{Area}(R_b) = 2\left(\frac{1}{2}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\right)$$

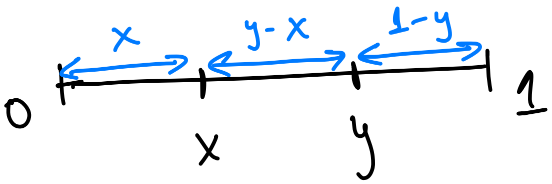
$$= \boxed{\frac{1}{2}} = 50\%$$



(Alternatively: $\text{Area}(R_b) = \frac{1}{2}\,\text{Area}(\Omega) = \frac{1}{4}$
"Doubling R_b we get all of Ω")

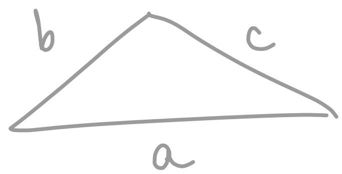
A very classical problem: If you break a stick at 2 points at random (chosen uniformly), what is the probability that the 3 resulting sticks form a triangle?

WLOG: Assume length is 1. Let $X, Y \sim \mathrm{Unif}([0,1])$



x, $y-x$, $1-y$ are the 3 sides of a triangle if and only if they satisfy the triangle inequality

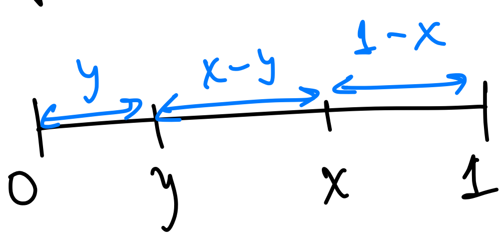
Recall: Triangle ineq.



$a < b + c$
$b < a + c$
$c < a + b$

$$\begin{cases} x < \cancel{y} - x + 1 - \cancel{y} \\ y - x < x + 1 - y \\ 1 - y < \cancel{x} + y - \cancel{x} \end{cases} \iff \begin{cases} x < 1/2 \\ y - x < 1/2 \\ y > 1/2 \end{cases}$$

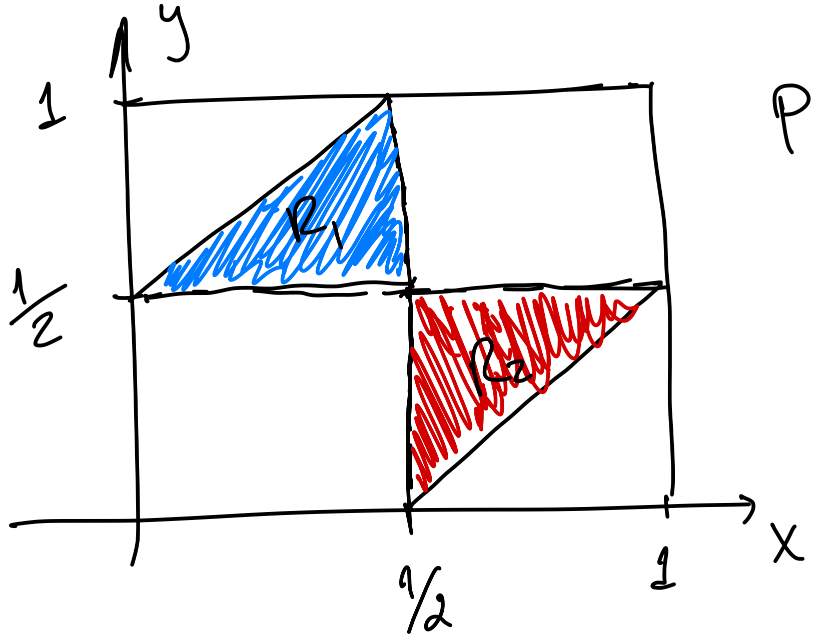
If the order is reversed:



$$\begin{cases} y < \cancel{x} - y + 1 - \cancel{x} \\ x - y < y + 1 - x \\ 1 - x < \cancel{y} + x - \cancel{y} \end{cases} \iff \begin{cases} y < 1/2 \\ x - y < 1/2 \\ x > 1/2 \end{cases}$$

3 resulting sticks form a triangle if and only if

$$\Big(\underbrace{x < \tfrac{1}{2} \text{ and } y - x < \tfrac{1}{2} \text{ and } y > \tfrac{1}{2}}_{R_1}\Big) \text{ or } \Big(\underbrace{y < \tfrac{1}{2} \text{ and } x - y < \tfrac{1}{2} \text{ and } x > \tfrac{1}{2}}_{R_2}\Big)$$



$$P\left(\begin{matrix}\text{can form}\\ \text{a triangle}\end{matrix}\right) = \iint_{R_1 \cup R_2} 1 \cdot dA$$

$$= \text{Area}(R_1 \cup R_2) = \frac{1}{4}$$

Ans: 25%

Independent Random Variables:

Recall: **Def**: X, Y are independent if $\forall A, B$,

$$P(X \in A, Y \in B) = P(X \in A)\, P(Y \in B).$$

Prop: Cont. rand. var. X, Y are independent if and only if their joint probability density function factors as

$$f_{X,Y}(x,y) = h(x)\, g(y), \quad \forall x, y \in \mathbb{R}.$$

Pf: If X, Y are indep.,

$$P(X \leq a, Y \leq b) = P(X \leq a)\, P(Y \leq b)$$

$$\underbrace{\frac{\partial^2}{\partial a \partial b} P(X \leq a, Y \leq b)}_{f_{X,Y}(a,b)} = \underbrace{\frac{\partial^2}{\partial a \partial b} P(X \leq a) P(Y \leq b)}_{\frac{\partial}{\partial a}\left(\frac{\partial}{\partial b} P(X \leq a) P(Y \leq b)\right)}$$

$$f_{X,Y}(a,b) = \frac{\partial}{\partial a} P(X \le a) \cdot \frac{\partial}{\partial b} P(Y \le b)$$
$$= f_X(a) \cdot f_Y(b), \quad \forall a,b \in \mathbb{R}$$

Conversely, suppose

$$f_{X,Y}(x,y) = h(x)\, g(y) \quad \forall x,y \in \mathbb{R}$$

Then

$$P(X \in A, Y \in B) = \iint_{A \times B} f_{X,Y}(x,y)\, dA$$
$$= \int_A \int_B h(x) g(y)\, dy\, dx$$
$$= \int_A h(x) \left(\int_B g(y)\, dy \right) dx$$
$$= \int_A h(x)\, dx \cdot \int_B g(y)\, dy$$
$$\overset{(*)}{=} P(X \in A) \cdot P(Y \in B).$$

$\square$

Claim:

(*) Here we are using that if $f_{X,Y}(x,y) = h(x) g(y)$, then, up to scaling $h(x) = f_X(x)$ and $g(y) = f_Y(y)$. This can be proven as follows:

If $f_{X,Y}(x,y) = h(x)\,g(y)$, we have

$$1 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx\,dy = \underbrace{\int_{-\infty}^{+\infty} h(x)\,dx}_{c_1} \cdot \underbrace{\int_{-\infty}^{+\infty} g(y)\,dy}_{c_2}$$

$$1 = c_1 \cdot c_2$$

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy = \int_{-\infty}^{+\infty} h(x)g(y)\,dy$$

$$= h(x) \cdot \underbrace{\int_{-\infty}^{+\infty} g(y)\,dy}_{c_2} = c_2 \cdot h(x)$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx = \int_{-\infty}^{+\infty} h(x)g(y)\,dx =$$

$$= g(y) \underbrace{\int_{-\infty}^{+\infty} h(x)\,dx}_{c_1} = c_1 \cdot g(y)$$

So

$$f_X(x) \cdot f_Y(y) = c_2\, h(x) \cdot c_1\, g(y) = h(x) \cdot g(x).$$

Therefore, up to scaling, h and g are the marginal p.d.f.'s of X and Y as claimed.

$\square$

Ex: Suppose the joint p.d.f. of X and Y is $f_{X,Y}(x,y)$.
Are X and Y independent?

a) $$f_{X,Y}(x,y) = \begin{cases} 6e^{-2x}e^{-3y}, & 0<x<\infty,\ 0<y<\infty \\ 0 & \text{otherwise} \end{cases}$$

b) $f_{X,Y}(x,y) = 24xy$, $\quad 0<x<1,\ 0<y<1,\ 0<x+y<1$

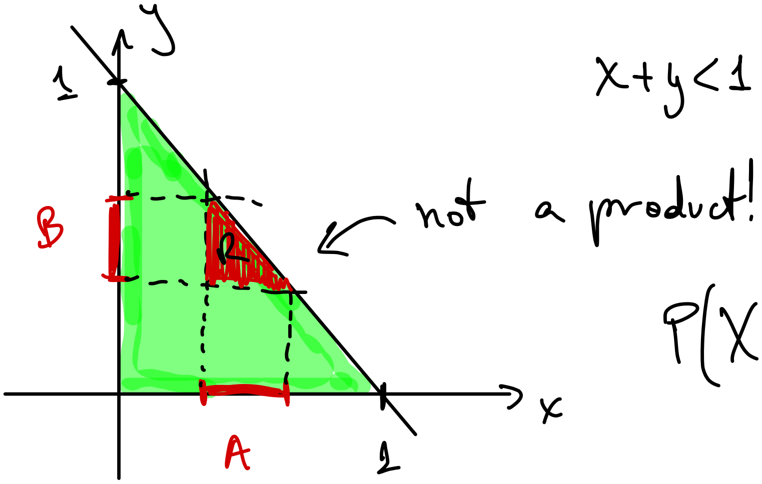
a) $f_{X,Y}(x,y) = \underbrace{6e^{-2x}}_{h(x)} \cdot \underbrace{e^{-3y}}_{g(y)}$ over the region $[0,+\infty) \times [0,+\infty)$

which is also a product

YES: X and Y are independent

b) $f_{X,Y}(x,y) = \underbrace{24x}_{h(x)}\,\underbrace{y}_{g(y)}$ but the region is not a product

NO: X and Y are not independent



$$x+y<1 \iff y<1-x$$

$$P(X\in A, Y\in B) = \iint_R f_{X,Y}(x,y)\,dx\,dy$$

$$P(X\in A) = \int_A f_X(x)\,dx = \int_A \underbrace{\left(\int_0^1 f_{X,Y}(x,y)\,dy\right)}_{f_X(x)} dx$$

$$P(Y\in B)=\int_B f_Y(y)\,dy=\int_B \underbrace{\left(\int_0^1 f_{X,Y}(x,y)\,dx\right)}_{f_Y(y)}dy$$

$$P(X\in A,\ Y\in B)\neq P(X\in A)\cdot P(Y\in B)$$