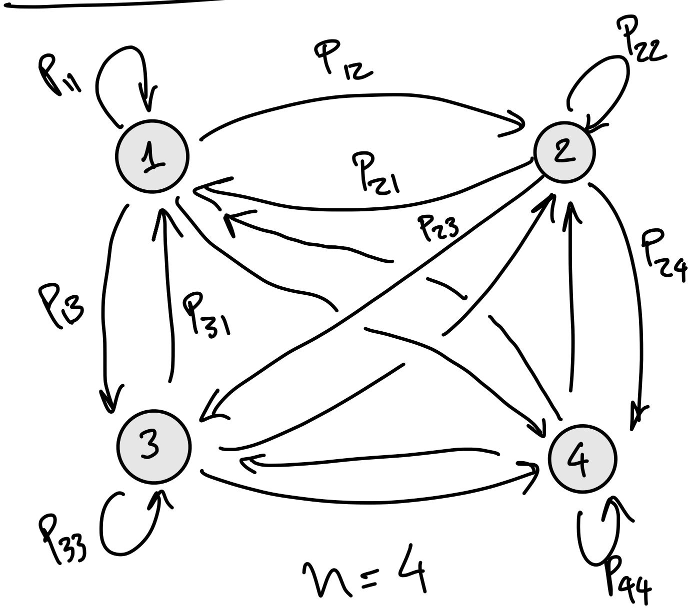
Markov Chains: Suppose that a system can be in one of n different states at each given time.



$n = 4$

state:	2	1	3	...	1 ...
Time:	1	2	3	...	k ...
random variables:	X_1	X_2	X_3	...	X_k ...

If the system is in state i, the probability that it transitions to state j in the next unit time is given by:

Transition probability from i to j.

$$P_{ij} = P(\underbrace{X_{k+1} = j}_{\text{next state being } j} \mid \underbrace{X_k = i}_{\text{current state is } i})$$

$\boxed{\text{Markov hypothesis}}$

$$\stackrel{\downarrow}{=} P(X_{k+1} = j \mid \underbrace{X_k = i}_{\text{current state}}, \underbrace{X_{k-1} = i_{k-1}, X_{k-2} = i_{k-2}, \ldots, X_0 = i_0}_{\text{past}}), \ \forall k.$$

Past is irrelevant to determine current transition probabilities.

Organize transition probabilities in a matrix, as follows:

$$
\begin{array}{r|ccccc|l}
 & \text{next state } 1 & \text{next state } 2 & \text{next state } 3 & & \text{next state } n & \\
 & 1 & 2 & 3 & \cdots & n & \\
\hline
\text{current state } 1 \to & P_{11} & P_{12} & P_{13} & \cdots & P_{1n} & 1 \\
\text{current state } 2 \to & P_{21} & P_{22} & P_{23} & \cdots & P_{2n} & 2 \\
P = \quad & P_{31} & P_{32} & P_{33} & \cdots & P_{3n} & 3 \\
 & \vdots & \vdots & \vdots & & \vdots & \vdots \\
\text{current state } n \to & P_{n1} & P_{n2} & P_{n3} & \cdots & P_{nn} & n
\end{array}
$$

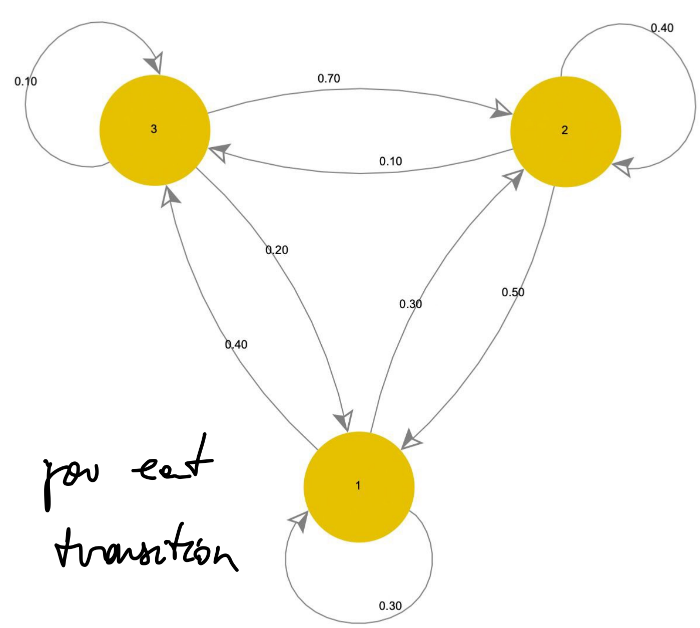
Clearly, the sum of probabilities in each line of P is 1.

$$\sum_{j=1}^{n} P_{ij} = 1.$$

this means that P is a "stochastic matrix."

Example. Suppose you eat 1 of 3 desserts every day:

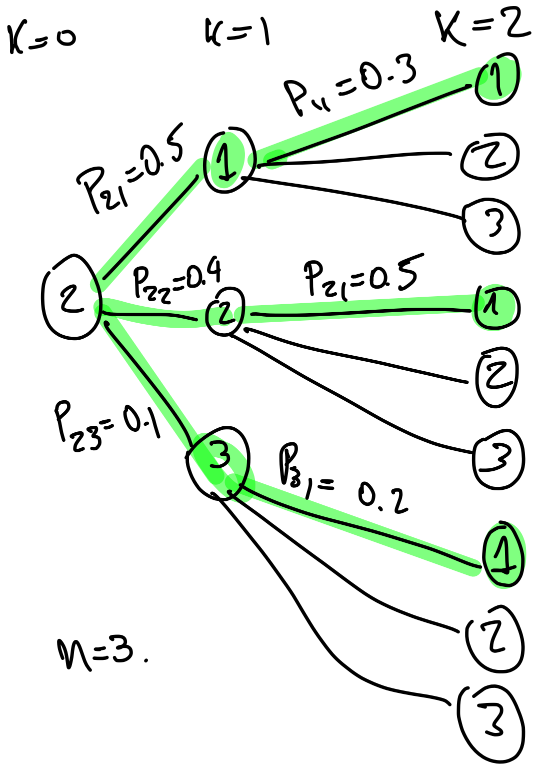
1. Ice Cream
2. Cake
3. Chocolate



Assume the sequence of desserts you eat form a Markov chain with transition probabilities as indicated.

$$P = \begin{pmatrix} 0.3 & 0.3 & 0.4 \\ 0.5 & 0.4 & 0.1 \\ 0.2 & 0.7 & 0.1 \end{pmatrix}$$

Q: What is the probability that you eat Ice Cream in <u>2 days</u>, given that you ate Cake today?



$$P^{(2)}_{21} := P(X_2 = 1 \mid X_0 = 2) = ?$$

$$P^{(2)}_{ij} = \sum_{k=1}^{n} P_{ik}P_{kj} \leftarrow \text{entry } (i,j) \text{ of matrix } P^2$$

$$P^{(2)}_{21} = \overbrace{P_{21} \cdot P_{11} + P_{22} \cdot P_{21} + P_{23} \cdot P_{31}}$$

$$= 0.5 \cdot 0.3 + 0.4 \cdot 0.5 + 0.1 \cdot 0.2$$

$$= 0.15 + 0.20 + 0.02$$

$$= 0.37$$

$$= 37\%$$

More generally, $P^{(N)}_{ij} = (i,j)$ entry of matrix P^N.

Q: What is the probability that you eat Ice Cream in <u>200 days</u>, given that you ate Cake today?

<u>Using Linear Algebra</u>,

$$P^{200} \cong \begin{pmatrix} 0.36 & 0.43 & 0.21 \\ \boxed{0.36} & 0.43 & 0.21 \\ 0.36 & 0.43 & 0.21 \end{pmatrix}$$

$N = 200$

$$P^{(200)}_{21} \cong 0.36 = 36\%$$

Chapman-Kolmogorov Equation

$$P_{ij}^{(N)} = \sum_{k=1}^{n} P_{ik}^{(r)} P_{kj}^{(N-r)}, \quad \text{for any } 0<r<N.$$

Probability of going from state i to state j after N steps.

$\left(P_{ij}^{(1)} = P_{ij} \right)$

In particular, if $r=1$, then

$$P_{ij}^{(N)} = \sum_{k=1}^{n} P_{ik} \cdot P_{kj}^{(N-1)}$$

(Recursively defining the (i,j)-entry of P^N.)

Pf. $P_{ij}^{(N)} = P(X_N = j \mid X_0 = i)$

$$= \sum_{k=1}^{n} \underbrace{P(X_N=j, X_r = k \mid X_0 = i)}_{\frac{P(X_N=j,\, X_r=k,\, X_0=i)}{P(X_0=i)}}$$

$$= \sum_{k=1}^{n} \underbrace{P(X_N=j \mid X_r=k, X_0=i)}_{\frac{P(X_N=j,\, X_r=k,\, X_0=i)}{P(\cancel{X_r=k,\, X_0=i})}} \cdot \overbrace{\underbrace{P(X_r=k \mid X_0=i)}_{\frac{P(\cancel{X_r=k,\, X_0=i})}{P(X_0=i)}}}^{P_{ik}^{(r)}}$$

By Markov hypothesis, $P(X_N=j \mid X_r=k, X_0=i) = P(X_N=j \mid X_r=k) = P_{kj}^{(N-r)}$

$$\cdots = \sum_{k=1}^{n} P_{kj}^{(N-r)} \cdot P_{ik}^{(r)} = \sum_{k=1}^{n} P_{ik}^{(r)} \cdot P_{kj}^{(N-r)}.$$

Q1: Over the long run, with what frequency is state j visited?

Q2: As time goes to $+\infty$, in what state is the system most likely to be?

Q1': $\pi_j := \lim_{N\to\infty} P_{ij}^{(N)} = ?$ Does it exist? How to compute it?

Q2': Which π_j, $1 \le j \le n$, is the largest?

Def: A Markov chain is ergodic if there exists $N > 0$ s.t. all entries of P^N are strictly positive, i.e., $P_{ij}^{(N)} > 0$, for all $1 \le i, j \le n$.

(i.e. one can reach every state from any other state within N steps.)

Thm: If a Markov chain is ergodic, then the limits $\pi_j = \lim_{N\to\infty} P_{ij}^{(N)}$ exist for all $1 \le j \le n$, and can be computed as the unique non-negative solutions to

$$\sum_{j=1}^{n} \pi_j = 1 \quad \text{and} \quad \sum_{k=1}^{n} \pi_k P_{kj} = \pi_j.$$

i.e. $P^t \vec{\pi} = \vec{\pi}$ is an eigenvector of P^t w/ eigenvalue 1, whose entries are ≥ 0 and add up to 1.

In the example from before, with transition matrix

$$P = \begin{pmatrix} 0.3 & 0.3 & 0.4 \\ 0.5 & 0.4 & 0.1 \\ 0.2 & 0.7 & 0.1 \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix}$$

we have:

- Markov chain is <u>ergodic</u> $P_{ij} > 0, \ 1 \leq i,j \leq 3.$
- π_j can be computed as the nonnegative solution to:

$$\vec{\pi} = (\pi_1, \pi_2, \pi_3)$$

$$\begin{cases} \pi_1 + \pi_2 + \pi_3 = 1 \quad \longleftarrow \langle \vec{\pi}, \vec{1} \rangle = 1 \\ \left.\begin{aligned} \pi_1 &= \pi_1 P_{11} + \pi_2 P_{21} + \pi_3 P_{31} \\ \pi_2 &= \pi_1 P_{12} + \pi_2 P_{22} + \pi_3 P_{32} \\ \pi_3 &= \pi_1 P_{13} + \pi_2 P_{23} + \pi_3 P_{33} \end{aligned}\right\} \quad \boxed{P^t \vec{\pi} = \vec{\pi}} \end{cases}$$

In other words, using <u>Linear Algebra</u>, the above is equivalent to finding an eigenvector $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ for the matrix P^t with eigenvalue 1, whose entries are nonnegative and add up to 1.

$$\vec{\pi} = (\pi_1, \pi_2, \pi_3) = \left(\frac{47}{129}, \frac{55}{129}, \frac{9}{43} \right) \doteq (0.36, 0.43, 0.21)$$

(cf. with entries of P^{200}.)