Random Variables

Def: $X : \Omega \longrightarrow \mathbb{R}$ (measurable) function is a random variable.

sample space

Ex 1: Throw 2 dice

$$\Omega = \{(1,1), (1,2), \dots, (1,6),$$
$$(2,1), (2,2), \dots, (2,6)$$
$$\vdots$$
$$(6,1), (6,2), \dots, (6,6)\}$$

Sample space: 36 possible outcomes.

$X : \{(1,1), (1,2), \dots, (6,6)\} \longrightarrow \mathbb{R}$ sum of the results

$$X\big((a,b)\big) = a + b.$$

$$X(1,1) = 2$$
$$X(1,2) = 3 \quad \dots \quad X(6,6) = 12.$$

$P\left(\boxed{X = 5}\right) = \frac{4}{36}$; $\quad P(X=12) = \frac{1}{36}$; $\quad P(X=10) = \frac{3}{36}$

This defines an event! Namely, the subset of Ω given by $X^{-1}(5)$

$1+4=5$
$2+3=5$
$3+2=5$
$4+1=5$

$6+6=12$

$6+4=10$
$5+5=10$
$4+6=10$

$$P(X \geq 10) = P(X=10) + P(X=11) + P(X=12)$$

$$= \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{6}{36} = \frac{1}{6}$$

$$P(X<10) = 1 - P(X \geq 10) = 1 - \frac{1}{6} = \frac{5}{6}$$

$$\boxed{\sum_{n=2}^{12} P(X=n) = 1}$$

$$P(X=100) = 0.$$

Ex2: 1 2 3 - - 20

20 balls
4 chosen at random
w/o replacement

$$\Omega = \{(1,2,3,4), (1,2,3,5), \ldots, (17,18,19,20)\}$$

Sample space has $\binom{20}{4}$ possible outcomes.

$X : \Omega \to \mathbb{R}$ largest number.

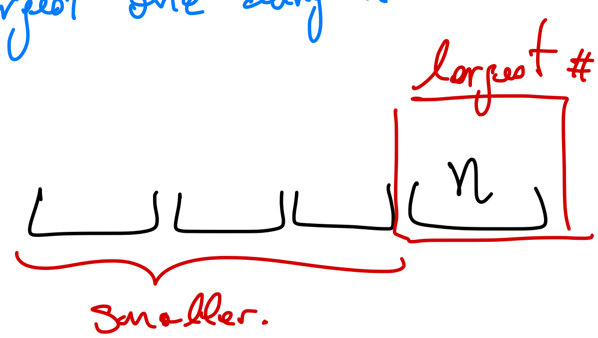
$X(1,2,3,4) = 4$ $\qquad$ $X(5,7,12,13) = 13$

Image $X = \{4, 5, 6, \ldots, 20\}$

$$P(X=n) = \frac{\binom{1}{1} \cdot \binom{n-1}{3}}{\binom{20}{4}}$$

selections of 4 balls, with largest one being n.

$|\Omega|$



$\forall n \in$ Image $X = \{4, \ldots 20\}$

$$P(X=10) = \frac{\binom{1}{1}\binom{9}{3}}{\binom{20}{4}} = \frac{28}{1615} \cong 1.73\%$$

$$P(X>10) = \sum_{k=11}^{20} P(X=k) = \sum_{k=11}^{20} \frac{\binom{k-1}{3}}{\binom{20}{4}}$$

1, — 10

$\cancel{11, 12, 13, \ldots 20}$

$$= 1 - P(X \leq 10) = 1 - \frac{\binom{10}{4}}{\binom{20}{4}}$$

<u>Ex3</u>: Flip a biased coin: $P(H) = p$, $P(T) = 1-p$.

until you get <u>H</u> or <u>n times</u>.

$$\Omega = \{H, TH, TTH, T\ldots TH, \ldots, \underbrace{T\ldots}_{n-1}TH, \underbrace{T\ldots T}_{n}\}$$

$X : \Omega \to \mathbb{R}$ is the number of flips.

$$X(H) = 1,\ X(TH) = 2, \ldots, X(\underbrace{T\ldots T}_{n}) = n.$$

$$P(X=1) = p.$$

$$P(X=2) = (1-p)p$$

$$P(X=3) = (1-p)^2 p$$

$$P(X=i) = \begin{cases} (1-p)^{i-1} p, & i = 1, \ldots, n-1. \\ (1-p)^{n-1}, & i = n \end{cases}$$

↳ 2 ways this can happen: $\underbrace{TTT\ldots T}_{n}$ or $\underbrace{T\ldots T}_{n-1}\underset{n}{H}$

$$= (1-p)^n + (1-p)^{n-1} p.$$

$$= (1-p)(1-p)^{n-1} + (1-p)^{n-1} p$$

$$= (1-p)^{n-1}(1-p+p) = (1-p)^{n-1}$$

"Sanity check"/"Consistency check":

$$\sum_{i=1}^{n} P(X=i) = 1.$$

$$\sum_{i=1}^{n-1} P(X=i) + P(X=n) = p \cdot \underbrace{\sum_{i=1}^{n-1} (1-p)^{i-1}}_{} + (1-p)^{n-1}$$

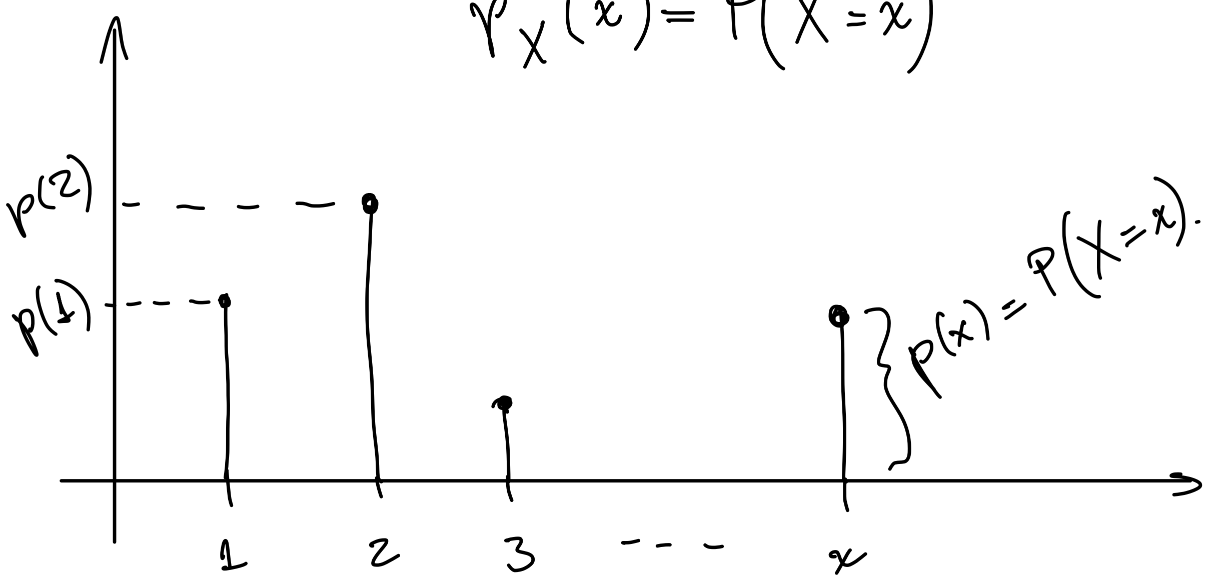
Finite sum (partial sum) of a geom. series

$$= p \cdot \frac{1-(1-p)^{n-1}}{\underbrace{1-(1-p)}_{p}} + (1-p)^{n-1}$$

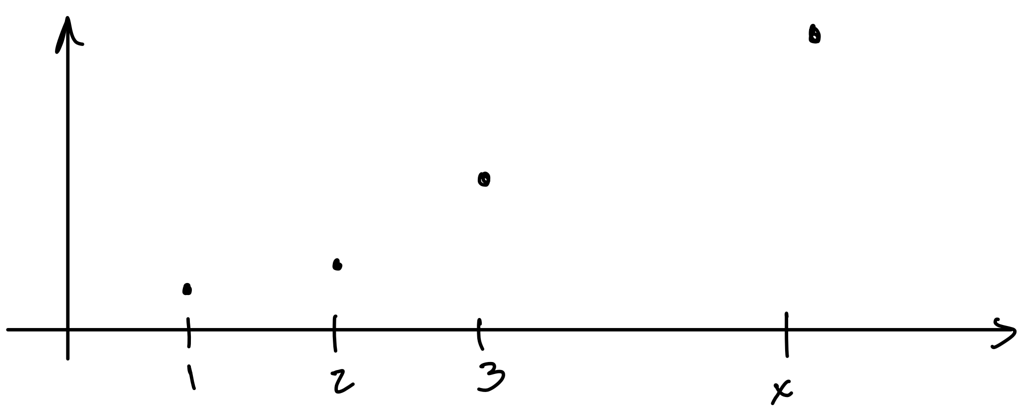
$$= 1-(1-p)^{n-1} + (1-p)^{n-1} = 1$$

Def: The probability mass function of the random variable $X: \Omega \to \mathbb{R}$ is

$$p_X(x) = P(X=x)$$



Def: (Cumulative) distribution function $F_X(x) = P(X \le x)$



Def: The <u>expected value</u> of $X: \Omega \to \mathbb{R}$ is

$$E(X) = \sum_x x \cdot \underbrace{p_X(x)}$$

Weighted average of values assumed by X, weighted by the probabilities that they are assumed.

$\boxed{p_X(x) = P(X=x) \quad \text{Prob. mass function}}$

Revisit the examples:

Roll 1 die:

$\Omega = \{1,2,3,4,5,6\}$

$X: \Omega \to \mathbb{R}$ outcome

$X(i) = i. \quad p_X(x) = \frac{1}{6}$

$$E(X) = 1\cdot\frac{1}{6} + 2\cdot\frac{1}{6} + 3\cdot\frac{1}{6} + 4\cdot\frac{1}{6} + 5\cdot\frac{1}{6} + 6\cdot\frac{1}{6}$$

$$= \frac{1+2+3+4+5+6}{6} = \frac{21}{6} = \frac{7}{2} = 3.5.$$

Ex 1: X = sum of results of rolling 2 dice.

$\Omega = \{(1,1), (1,2), \ldots, (6,6)\}$

$$E(X) = 2\cdot\underbrace{P(X=2)}_{p_X(2)} + 3\cdot\underbrace{P(X=3)}_{p_X(3)} + 4\cdot\underbrace{P(X=4)}_{p_X(4)} + \cdots + 12\cdot\underbrace{P(X=12)}_{p_X(12)}$$

$$= 2\cdot\frac{1}{36} + 3\cdot\frac{2}{36} + 4\cdot\frac{3}{36} + \cdots + 12\cdot\frac{1}{36} \overset{\text{some computation}}{=} \frac{252}{36} = 7.$$

Ex 2.

$$p_X(n) = P(X=n) = \frac{\binom{1}{1}\cdot\binom{n-1}{3}}{\binom{20}{4}} = \frac{\binom{n-1}{3}}{\binom{20}{4}}$$

$$E(X) = \sum_n n\cdot p_X(n) = \sum_{n=4}^{20} n\cdot\frac{\binom{n-1}{3}}{\binom{20}{4}} =$$

$$= 4\,\underbrace{\frac{\binom{3}{3}}{\binom{20}{4}}}_{n=4} + 5\cdot\underbrace{\frac{\binom{4}{3}}{\binom{20}{4}}}_{n=5} + \cdots + 20\cdot\underbrace{\frac{\binom{19}{3}}{\binom{20}{4}}}_{n=20} \overset{\text{some computation}}{=} \frac{84}{5} = \underline{\underline{16.8}}.$$

Ex 3:

$$p_X(i) = P(X=i) = \begin{cases} (1-p)^{i-1}\,p, & i=1,\dots,n-1. \\ (1-p)^{n-1}, & i=n \end{cases}$$

$$E(X) = \sum_{i=1}^{n} i\cdot p_X(i) = \sum_{i=1}^{n-1} (1-p)^{i-1}\cdot p\cdot i + n\cdot(1-p)^{n-1}$$

$$= \underset{\uparrow\, i=1}{p} + \underset{\uparrow\, i=2}{2\cdot p\cdot(1-p)} + \underset{\uparrow\, i=3}{3\cdot p(1-p)^2} + \cdots + \underset{\uparrow\, i=n.}{\underbrace{n(1-p)^{n-1}}}$$

$$\boxed{\dots = \frac{1-(1-p)^n}{p}}$$

Example: a) Flip a coin: $P(H)=\frac{1}{2}$, $P(T)=\frac{1}{2}$

- You win \$5 if H
- You lose \$4 if T

X = income from the game, $X:\{H,T\}\to\mathbb{R}$

$X(H)=5$, $X(T)=-4$.

$$E(X) = (+5)\cdot\underset{\substack{\parallel\\ P(H)\\ \parallel\\ P(X=5)}}{\frac{1}{2}} + (-4)\cdot\underset{\substack{\parallel\\ P(T)\\ \parallel\\ P(X=-4)}}{\frac{1}{2}} = \frac{1}{2} = 0.5 > 0$$

b) What if the coin is biased?

$$P(H) = \frac{1}{3} \qquad P(T) = \frac{2}{3}.$$

$$E(X) = (+5)\cdot \underbrace{\frac{1}{3}}_{= P(X=5)} + (-4)\cdot \underbrace{\frac{2}{3}}_{= P(X=-4)} = \frac{5-8}{3} = -1 < 0$$