Recap on eigenvalues and eigenvectors:

- $T: \mathbb{R}^n \to \mathbb{R}^n$ (or $T: V \to V$)

  $v \in \mathbb{R}^n$ is an eigenvector of T if $v \neq 0$ and $T(v) = \lambda v$ for some $\lambda \in \mathbb{R}$.

  eigenvalue

- To find eigenvalues, solve the characteristic equation $\det(T - \lambda \mathrm{Id}) = 0$. Then, to find corresponding eigenvectors, find a basis of $\underbrace{E_\lambda = \ker(T - \lambda \mathrm{Id})}_{\lambda\text{-eigenspace}}$.

Ex: $T = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$ $\qquad T: \mathbb{R}^3 \to \mathbb{R}^3$

For example, expand here to compute determinant

$$T - \lambda \mathrm{Id} = \begin{pmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{pmatrix}$$

$$0 = \det(T - \lambda \mathrm{Id}) = (1-\lambda)\det\begin{pmatrix} -5-\lambda & -3 \\ 3 & 1-\lambda \end{pmatrix}$$

$$- 3 \det\begin{pmatrix} -3 & -3 \\ 3 & 1-\lambda \end{pmatrix}$$

$$+ 3 \det\begin{pmatrix} -3 & -5-\lambda \\ 3 & 3 \end{pmatrix}$$

$$= (1-\lambda)\left[(-5-\lambda)(1-\lambda)+9\right]$$
$$-3\left[-3(1-\lambda)+9\right] + 3\left[-9+(5+\lambda)\cdot 3\right]$$
$$= (1-\lambda)\left(-5+5\lambda-\lambda+\lambda^2+9\right)$$
$$\cancel{-3\left(3\lambda+6\right)} + \cancel{3\left(3\lambda+6\right)}$$
$$= (1-\lambda)\left(\lambda^2+4\lambda+4\right)$$
$$= (1-\lambda)(\lambda+2)^2.$$

double root of the equation

<u>Characteristic equation</u>: $(\lambda+2)^2(\lambda-1) = 0$

Solutions are $\begin{cases} \lambda_2 = -2 & \text{algebraic multiplicity } 2 \\ \lambda_1 = 1 & \text{algebraic multiplicity } 1 \end{cases}$

Look for bases of $E_{-2} = \mathrm{Ker}(T+2\,\mathrm{Id})$, $E_1 = \mathrm{Ker}(T-\mathrm{Id})$.

<u>E_{-2}:</u> $T + 2\,\mathrm{Id} = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix} + 2\,\mathrm{Id} = \begin{pmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{pmatrix}$

$\overset{R.R.}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{cases} x_1 = -x_2 - x_3 \\ x_2 \\ x_3 \quad \text{free} \end{cases}$

$\left.\begin{matrix} x_2 = 1 \\ x_3 = 0 \end{matrix}\right\} \Rightarrow x_1 = -1 \qquad \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad \left.\begin{matrix} x_2 = 0 \\ x_3 = 1 \end{matrix}\right\} \Rightarrow x_1 = -1 \qquad \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$$E_{-2} = \mathrm{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

basis for this eigenspace.

$\underline{E_1}$:

$$T - Id = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix} - Id = \begin{pmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{pmatrix} \sim$$

$$\overset{R.R.}{\sim} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{cases} x_1 = -x_2 \\ x_2 = -x_3 \\ x_3 \text{ free} \end{cases} \quad x_3 = 1 \quad \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$E_1 = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

basis for this eigenspace.

Diagonalizing a matrix / linear transformation

$$T = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix} \overset{\text{diagonalize}}{\rightsquigarrow} D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

original / input

diagonal / output

unique up to reordering the diagonal entries

Def: The matrix / linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is <u>diagonalizable</u> if there exists an <u>isomorphism $P: \mathbb{R}^n \to \mathbb{R}^n$</u> such that $T = PDP^{-1}$ where D is diagonal.

invertible linear transf.

Note: $\boxed{T = PDP^{-1}} \iff P^{-1}TP = \underbrace{P^{-1}P}_{=Id} D \underbrace{P^{-1}P}_{=Id} = D$

"conjugation by P" $\iff \boxed{P^{-1}TP = D}$

From previous video:

basis for $E_1 : \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

basis for $E_{-2} : \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow P = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

"change of basis"

Check that:

$$D = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}}_{\text{output}} = P^{-1} \underbrace{\begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}}_{\text{input}} P$$

Some matrices are <u>not</u> diagonalizable:

$$T = \begin{pmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{pmatrix} \qquad \det(T - \lambda \mathrm{Id}) = \underbrace{(1-\lambda)(\lambda+2)^2}$$

same as in the previous example!

$\lambda_1 = 1$ w/ multiplicity 1

$\lambda_2 = -2$ w/ multiplicity 2

Now, find bases for E_1 and E_{-2}.

<u>E_1:</u> $T - \mathrm{Id} = \begin{pmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 4 & 3 \\ -4 & -7 & -3 \\ 3 & 3 & 0 \end{pmatrix} \overset{\text{R.R.}}{\sim} \begin{pmatrix} 1 & 4 & 3 \\ 1 & 1 & 0 \\ 0 & -3 & -3 \end{pmatrix} \overset{\text{R.R.}}{\sim} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\rightsquigarrow \begin{cases} x_1 = -x_2 \\ x_2 = -x_3 \\ x_3 \quad \text{free} \end{cases}$$

Choose $x_3 = 1$:

$$E_1 = \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

$\underline{E_{-2}}$:

$$\begin{pmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{pmatrix} + 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{pmatrix} \overset{\text{R.R.}}{\sim}$$

$$\sim \begin{pmatrix} 1 & 1 & 3/4 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3/4 \\ 0 & 0 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}$$

How many free variables?

just 1: x_2 free

$$\rightsquigarrow \begin{cases} x_1 = -x_2 - \frac{3}{4} x_3 = -x_2 \\ x_2 \quad \text{free} \\ x_3 = 0 \end{cases}$$

Choose $x_2 = 1$:

$$E_{-2} = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Putting all eigenvectors together: $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

No third one!

This T is <u>not</u> diagonalizable!

The issue is that

$$\dim E_{-2} = 1 < 2 = \text{mult}(-2)$$

(even though: $\dim E_1 = 1 = \text{mult}(1)$)

Theorem. An $n \times n$ matrix whose distinct real eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_p$ is diagonalizable if and only if

$$\underbrace{\dim E_{\lambda_j}}_{\substack{\text{dimension of} \\ \text{the } \lambda_j\text{-eigenspace} \\ \text{"geometric multiplicity"}}} = \underbrace{\text{mult}(\lambda_j)}_{\substack{\text{number of} \\ \text{times that } \lambda_j \\ \text{appears as a} \\ \text{root of characteristic} \\ \text{equation} \\ \text{"algebraic multiplicity"}}} \quad \text{and} \quad \underbrace{\sum_{j=1}^{p} \text{mult}(\lambda_j) = n.}_{\substack{\text{the characteristic} \\ \text{equation factors} \\ \text{completely as} \\ (\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_p)^{m_p} \\ 1 \le m_j = \text{mult}(\lambda_j) \\ \text{w/ } \lambda_j \in \mathbb{R}.}}$$

In general, $\dim E_{\lambda_j} \le \text{mult}(\lambda_j)$.

- Ways in which matrices can **fail** to be diagonalizable:

Ex 1:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Characteristic equation:

$\det(A - \lambda \mathrm{Id}) = -\lambda^3 = 0$

has solution $\lambda_1 = 0$.

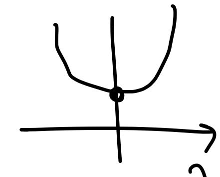
$E_0 = \operatorname{Ker} A = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$

$$\begin{cases} x_2 = 0 \\ x_3 = 0 \\ x_1 \text{ free} \end{cases} \quad \overset{x_1 = 1}{\rightsquigarrow} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\dim E_0 = 1 < 3 = \text{mult}(0)$$

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$B : \mathbb{R}^2 \to \mathbb{R}^2$



Characteristic equation:

$\det(B - \lambda \mathrm{Id}) = \lambda^2 + 1 > 0$

has <u>no real roots</u>!

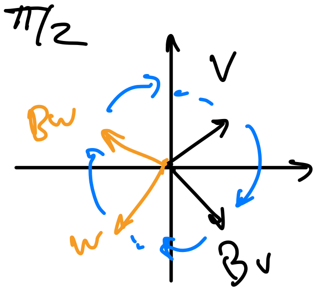
$$\sum_{j=1}^{p} \text{mult}(\lambda_j) = 0 < 2 = n$$

Side note: $\lambda = \pm i$ are complex eigenvalues

<u>Geometric explanation:</u>

$B = R_\theta$ for $\theta = \pi/2$

rotation by $\pi/2$



• A $\underbrace{\text{way to guarantee that}}_{\text{"sufficient condition"}}$ a matrix is diagonalizable:

Prop: If A is an $n\times n$ matrix and it has n distinct eigenvalues, then A is diagonalizable.

Pf: If A has n distinct eigenvalues, then the char. eqn. factors as

$$\underbrace{\det(A-\lambda \mathrm{Id})}_{\text{polynomial of degree } n} = \underbrace{(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)}_{n \text{ distinct roots}}$$

Therefore $\mathrm{mult}(\lambda_j)=1$, so $\sum_{j=1}^{n}\mathrm{mult}(\lambda_j)=n$.

$1\leq \dim E_{\lambda_j}\leq \mathrm{mult}(\lambda_j)=1$, so $\dim E_{\lambda_j}=\mathrm{mult}(\lambda_j)$,

thus A is diagonalizable by Theorem above. $\square$

Ex:

$$A=\begin{pmatrix} 1 & 0 & 3 & 41 & e & 1\\ 0 & 2 & 4 & \pi & 1/2 & 1\\ 0 & 0 & -1 & 5 & 15 & 0\\ 0 & 0 & 0 & -7 & 8 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 4\end{pmatrix}$$

6×6 matrix

Q: Is A diagonalizable?

A: Yes! The matrix A is diagonalizable because its eigenvalues are $1, 2, -1, -7, 0, 4$, which are 6 distinct eigenvalues.

Note: $\det(A-\lambda \mathrm{Id}) = \underbrace{(1-\lambda)(2-\lambda)(-1-\lambda)(-7-\lambda)(-\lambda)(4-\lambda)}_{\text{all distinct roots!}}$

Why would we want to diagonalize a matrix?

Reason 1: "Geometrically recognize" how a linear transformation acts on the space

Reason 2: Singular Value Decomposition, Principal Component Analysis Machine learning, models of Markov chains, etc.

to be discussed more in a future class!

Reason 3$^{\circledast}$: To facilitate computation of powers of matrices.

If the matrix is diagonalizable.

$$A = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix} \qquad A^8 = ? \qquad A^{2022} = ?$$

$$A^2 = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 10 & -9 \\ 6 & -5 \end{pmatrix}$$

$$A^3 = \underbrace{\begin{pmatrix} 10 & -9 \\ 6 & -5 \end{pmatrix}}_{A^2}\begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 22 & -21 \\ 14 & -13 \end{pmatrix}$$

$\vdots$

There's a better way!

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad D^2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}$$

$$D^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}$$

Diagonalizing A we have $A = PDP^{-1}$

$$A^8 = \underbrace{(PD\cancel{P^{-1}})}_{A}\underbrace{(\cancel{P}D\cancel{P^{-1}})}_{A}\cancel{\cdots}\underbrace{(\cancel{P}DP^{-1})}_{A} = PD^8P^{-1}$$

all P's and (P^{-1})'s cancel

So let's diagonalize $A = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}$!

Characteristic equation:

$$\det(A - \lambda \mathrm{Id}) = \det\begin{pmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{pmatrix} = (4-\lambda)(-1-\lambda) + 6$$

$$= -4 - 4\lambda + \lambda + \lambda^2 + 6 = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1)$$

Eigenvalues are $\underline{2}$ and $\underline{1}$, both have multiplicity 1.

Find bases for the eigenspaces:

$$E_2 = \mathrm{Ker}(A - 2\mathrm{Id}) = \mathrm{span}\left\{\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right\}$$

$$A - 2\mathrm{Id} = \begin{pmatrix} 2 & -3 \\ 2 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -3/2 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{cases} x_1 = 3/2\, x_2 \\ x_2 \text{ free} \end{cases}$$

choose $x_2 = 2$

$$v_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$E_1 = \mathrm{Ker}(A - \mathrm{Id}) = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$

$$A - \mathrm{Id} = \begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{cases} x_1 = x_2 \\ x_2 \text{ free} \end{cases}$$

choose $x_2 = 1$

$$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \quad \text{is such that} \quad A = PDP^{-1}$$

where $D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

find inverse

$$\left(P \;\middle|\; Id \right) \overset{R.R.}{\sim} \cdots \sim \left(Id \;\middle|\; \boxed{P^{-1}} \right) \qquad P^{-1} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

$$A^8 = PD^8P^{-1} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \overset{256}{\overbrace{2^8}} & 0 \\ 0 & \underset{1}{\underbrace{1^8}} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 766 & -765 \\ 510 & -509 \end{pmatrix}$$

$$A^{2022} = PD^{2022}P^{-1} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2^{2022} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

Capstone: Relating diagonalization w/ change of basis

$\mathcal{E}$: "canonical basis"

diagonalize

$\mathcal{B}$: "basis of eigenvectors"

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "T"', "\text{mult. by } {[T]_{\mathcal{E}}}"] \arrow[d, "P^{-1}"] & \mathbb{R}^n \\
\mathbb{R}^n \arrow[r, "T"', "\text{mult. by } {[T]_{\mathcal{B}}}"] & \mathbb{R}^n \arrow[u, "P"']
\end{tikzcd}$$

Note $T = [T]_{\mathcal{E}}$ and $[T]_{\mathcal{B}} = D$ is a diagonal matrix if $\mathcal{B}$ is formed by eigenvectors of T:

$$[T]_{\mathcal{B}} = \begin{pmatrix} | & & | \\ [Tb_1]_{\mathcal{B}} & \cdots & [Tb_n]_{\mathcal{B}} \\ | & & | \end{pmatrix}$$

Indeed, if $\mathcal{B} = \{b_1, \dots, b_n\}$ is formed by eigenvectors, then

$$T b_1 = \lambda_1 b_1, \quad T b_2 = \lambda_2 b_2, \quad \dots, \quad T b_n = \lambda_n b_n$$

$$[Tb_1]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad [Tb_2]_{\mathcal{B}} = \begin{pmatrix} 0 \\ \lambda_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$[T]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & & \lambda_n \end{pmatrix} = D \text{ is diagonal!}$$

w.r.t. canonical basis $\mathcal{C}$

Change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "T", "\text{mult. by } [T]_{\mathcal{C}}"'] & \mathbb{R}^n \\
\mathbb{R}^n \arrow[r, "T", "\text{mult. by } [T]_{\mathcal{B}}"'] \arrow[u, leftarrow, "P^{-1}"] & \mathbb{R}^n \arrow[u, "P"']
\end{tikzcd}$$

$$P = P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{pmatrix} | & & | \\ [b_1]_{\mathcal{C}} & \cdots & [b_n]_{\mathcal{C}} \\ | & & | \end{pmatrix}$$

$$P^{-1} = P_{\mathcal{B} \leftarrow \mathcal{C}}$$

with respect to basis $\mathcal{B}$ of eigenvectors $\Big) \leftarrow$ exists iff $[T]_{\mathcal{C}}$ is diagonalizable

$$[T]_{\mathcal{C}} = P_{\mathcal{C} \leftarrow \mathcal{B}} \, [T]_{\mathcal{B}} \, P_{\mathcal{B} \leftarrow \mathcal{C}}$$

$$A = P \quad D \quad P^{-1}$$