MAT313, Fall 2025

Practice Problems for the Final Exam

1. Find the unique polynomial $p(t) = a_2t^2 + a_1t + a_0$ with $p(-1) = -4$, $p(1) = 2$, $p(2) = -1$.

   Answer: $p(t) = -2t^2 + 3t + 1$.

2. Find all solutions to the system of linear equations

$$\begin{cases} x_1 + 4x_2 - 3x_3 + 2x_4 = 0 \\ 2x_1 - 6x_2 + x_3 + x_4 = 0 \end{cases}$$

   Answer: $x_1 = x_3 - \frac{8}{7}x_4,\ x_2 = \frac{1}{2}x_3 - \frac{3}{14}x_4,\ x_3$ free, x_4 free.

3. Let $T\colon P_2 \to \mathbb{R}^2$ be given by $T(p) = \big(p(1),\, p(-1)\big)$. Determine whether T is a linear transformation. If so, then:

   (a) Find the matrix that represents T, where the basis of P_2 is $\{1, t, t^2\}$ and the basis of $\mathbb{R}^2$ is the standard basis $\{e_1, e_2\}$

   (b) Is T surjective?

   (c) Is T injective?

   (d) If T is bijective, then find its inverse.

   Answer: T is linear.

   a) $T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

   b) Yes: $\operatorname{Im} T = \mathbb{R}^2$

   c) No: $\dim\ker T = 1$ by the Rank–Nullity Theorem

   d) Not bijective

4. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(x, y, z) = (x + 3y - z,\ y + z,\ x + 7y + z)$. Determine whether T is a linear transformation. If so, then:

   (a) Find the matrix that represents T with respect to the standard basis $\{e_1, e_2, e_3\}$

   (b) Is T surjective?

   (c) Is T injective?

   (d) If T is bijective, then find its inverse.

Answer: a) The matrix that represents T in the canonical basis is

$$\begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 1 \\ 1 & 7 & 1 \end{bmatrix}$$

b) Yes: row reduction of above matrix has pivots in every column.

c) Yes: since $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is linear and surjective, it is also injective, by Rank–Nullity Theorem.

d) Yes, it is invertible and T^{-1} is represented in the canonical basis by the matrix

$$\begin{bmatrix} 3 & 5 & -2 \\ -\frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 2 & -\frac{1}{2} \end{bmatrix}$$

5. Let $T\colon \mathbb{R}^4 \to \mathbb{R}^2$ be a linear map, such that

$$T(e_1) = (-1, 9),\ T(e_2) = (8, 2),\ T(e_3) = (3, -1),\ T(e_4) = (1, 1).$$

   (a) Compute $T(v)$ where $v = (1, 2, 0, 1)$.
   (b) What is the matrix that represents T in the canonical basis?
   (c) What are the dimensions of the subspaces $\ker T$ and $\operatorname{Im} T$?

Answer: a) $T(v) = (16, 14)$,

b) $[T] = \begin{bmatrix} -1 & 8 & 3 & 1 \\ 9 & 2 & -1 & 1 \end{bmatrix}$

c) $\dim \operatorname{Im} T = 2, \quad \dim \ker T = 2$

6. Find all the values of $t \in \mathbb{R}$ such that the vectors $\left\{ \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} t \\ t-1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ t \end{bmatrix} \right\}$ are linearly dependent.

Answer: $t = 0,\ \ t = 1 + \sqrt{3},\ \ t = 1 - \sqrt{3}$.

7. Decide if each of the following sets is linearly dependent or linearly independent:

   (a) $\{(1, 3), (0, 1), (5, 7)\}$ in $\mathbb{R}^2$
   (b) $\{\sin t + \cos t,\ \sin t - \cos t\}$ in $C([0, 2\pi], \mathbb{R})$
   (c) $\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 2 \\ 2 & 2 \end{smallmatrix}\right)\}$ in $\mathrm{Mat}_{2\times 2}(\mathbb{R})$
   (d) $\{t^2 + 5t,\ t - 1,\ t,\ 7\}$ in P_3

Answer: a) lin dep, b) lin indep, c) lin dep, d) lin dep

8. Is $A = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$ invertible? If so, find its inverse.
   Answer: No, since $\det(A) = 0$.

9. Suppose that A is an invertible matrix. Is its transpose A^T also invertible? Justify.
   Answer: Yes, since $\det(A^T) = \det(A)$.

10. Suppose that A is symmetric. Is A invertible? Justify.
    Answer: No, e.g. $A = 0$ is symmetric but not invertible.

11. Suppose that A is symmetric. Is A^2 symmetric? Justify.
    Answer: Yes: $A^T = A$ so $(A^2)^T = (AA)^T = (A^T)(A^T) = AA = A^2$.

12. Suppose that A and B are invertible. Is A^2B^2 invertible? What about $ABAB$?
    Answer: Yes: $(A^2B^2)^{-1} = (B^{-1})^2(A^{-1})^2$, $(ABAB)^{-1} = B^{-1}A^{-1}B^{-1}A^{-1}$.

13. Compute the determinant of the matrices $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 0 \\ 5 & 4 & 4 \end{bmatrix}$.

    Answer: $\det A = 2, \ \det B = -12$

14. Find bases for $\ker T$ and $\operatorname{Im} T$ for each of the following linear transformations:

    (a) $T\colon \mathbb{R} \to \mathbb{R}^2$, $T(x) = (x, x)$
    (b) $T\colon P_2 \to P_3$, $T(p(t)) = \int_0^t p(s)\, \mathrm{d}s$
    (c) $T\colon \operatorname{Mat}_{2\times 2}(\mathbb{R}) \to \mathbb{R}^2$, $T(A) = (a_{11}, a_{22})$, where $A = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)$

    Answer:

    a) $\ker T = \{0\}, \quad \operatorname{Im} T = \operatorname{span}\{(1,1)\}$
    b) $\ker T = \{0\}, \quad \operatorname{Im} T = \operatorname{span}\{t, t^2, t^3\}$
    c) $\ker T = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}, \quad \operatorname{Im} T = \operatorname{span}\{(1,0), (0,1)\}$

15. Find the coordinates of $v = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ with respect to the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

    Answer: $[v]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

16. Find the eigenvalues of the following matrices

(a) $\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Answer:
a) $7, 1$
b) $2 \pm \sqrt{5}$
c) $2, 2, 0$

17. Find the eigenvalues and corresponding eigenspaces of

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

Answer: Eigenvalues: $\lambda = 5, 2$. Eigenspaces: $E_5 = \text{span}\{(1,1)\}$, $E_2 = \text{span}\{(-1,2)\}$.

18. Determine whether the matrix

$$A = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$$

is diagonalizable. If so, find a diagonalization.
Answer: Not diagonalizable because eigenvalue $\lambda = 4$ has algebraic multiplicity 2 but geometric multiplicity 1.

19. Determine whether the matrix

$$A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

is diagonalizable. If so, find a diagonalization.
Answer: Eigenvalues: $\lambda = 7, 2$. Eigenspaces: $E_7 = \text{span}\{(2,1)\}$, $E_2 = \text{span}\{(-1,2)\}$.
So $P = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix}$ are such that $A = PDP^{-1}$.

20. Find A^{100} where $A = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$.

Answer: $A^{100} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 7^{100} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{7^{100}+1}{2} & \frac{7^{100}-1}{2} \\ \frac{7^{100}-1}{2} & \frac{7^{100}+1}{2} \end{bmatrix}$.

21. Find A^{100} where $A = \begin{bmatrix} -5 & 4 \\ -6 & 5 \end{bmatrix}$.

    Answer: $A^{100} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

22. Apply Gram-Schmidt to find an orthonormal basis of $W = \text{span}\{(1, 1, 0), (1, 0, 1)\}$.

    Answer: Orthonormal basis: $\left\{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), (\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}})\right\}$

23. Find an orthonormal basis of the subspace $\{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$.

    Answer: Orthonormal basis: $\left\{(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0), (\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\sqrt{\frac{2}{3}})\right\}$

24. Find the point in the plane $x - 2y + z = 0$ which is closest to $(1, 0, 0)$.

    Answer: An orthonormal basis for the plane is $\left\{\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\right\}$, and the orthogonal projection of $(1, 0, 0)$ on this plane is $\left(\frac{5}{6}, \frac{1}{3}, -\frac{1}{6}\right)$.

25. Suppose you are a trader working for investors that want to have in their portfolio:

    - 4,000 shares of company A,
    - 3,000 shares of company B,
    - 2,000 shares of company C.

    However, you do not have access to individual shares of companies A, B, and C. Instead, you can only buy 2 types of combined investments:

    - 1 unit of investment type 1 has 1,000 shares of A and 1,000 shares of B,
    - 1 unit of investment type 2 has 1,000 shares of B and 1,000 shares of C.

    The investors request that if you cannot purchase exactly their desired portfolio, then you should purchase the combination of available investments which results in holdings that are as close as possible to their desired portfolio. How many units of investments type 1 and 2 do you purchase on their behalf?

    Answer: Orthogonal projection of $w = (4, 3, 2)$ onto the plane spanned by $v_1 = (1, 0, 1)$ and $v_2 = (0, 1, 1)$ is $\hat{w} = (3, 4, 1) = 3v_1 + v_2$, so 3 units of type 1 and 1 unit of type 2.

26. Let A be a 4×4 matrix with eigenvalues $-1, 0, 1, 2$. Is A invertible? If so, compute the eigenvalues of A^{-1}.

    Answer: No, A has nontrivial kernel (the eigenspace with eigenvalue 0), so it is not invertible. Note $\det A = (-1) \cdot 0 \cdot 1 \cdot 2 = 0$.

27. Let A be a 3×3 matrix with eigenvalues $2, 5, 10$. Is A invertible? If so, compute the eigenvalues of A^{-1}.

Answer: Yes, since $A = PDP^{-1}$ with P orthogonal and $D = \begin{bmatrix} 2 & & \\ & 5 & \\ & & 10 \end{bmatrix}$, we see that $\det A = 100$ so A is invertible. Moreover, $A^{-1} = (PDP^{-1})^{-1} = P^{-1}D^{-1}P$ so A^{-1} has eigenvalues $1/2, 1/5, 1/10$, with the same eigenvectors as A.

28. Let A be a 6×6 matrix with eigenvalues c and $-c$, where $c \neq 0$, such that the corresponding eigenspaces have dimension 3. Prove that $A^2 = c^2\,\mathrm{Id}$.

Answer: Since the eigenspaces have dimension 3, these eigenvalues have algebraic multiplicity ≥ 3. The matrix is 6×6, so the characteristic polynomial is of degree 6. Thus, the algebraic and geometric multiplicity of each eigenvalue is equal to 3 and the matrix is diagonalizable. Therefore, $A = PDP^{-1}$ for some invertible matrix P and D is diagonal with eigenvalues $\pm c$. Thus, $A^2 = PD^2P^{-1}$ has eigenvalues c^2 and $(-c)^2 = c^2$, which are equal, so $D^2 = c^2\mathrm{Id}$ and hence $A^2 = c^2P^{-1}P = c^2\,\mathrm{Id}$.

29. If A is diagonalizable, prove that $A + c\,\mathrm{Id}$ is also diagonalizable for any $c \in \mathbb{R}$.

Answer: Since $A = PDP^{-1}$ with P orthogonal and D diagonal, we see that

$$A + c\,\mathrm{Id} = PDP^{-1} + c\,\mathrm{Id} = PDP^{-1} + P\,c\,\mathrm{Id}\,P^{-1} = P(D + c\,\mathrm{Id})P^{-1}$$

is diagonalizable and has eigenvalues $\lambda + c$ where λ is an eigenvalue of A.

30. Does the matrix $\begin{bmatrix} 4 & 0 & -1 \\ 0 & 2 & 1 \\ 1 & -1 & 3 \end{bmatrix}$ admit an orthonormal basis of eigenvectors? Justify.

Answer: No, because it is not symmetric.

31. Suppose $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are vectors such that

$$\langle \vec{v}_1, \vec{v}_1 \rangle = 5, \quad \langle \vec{v}_1, \vec{v}_2 \rangle = 2, \quad \langle \vec{v}_1, \vec{v}_3 \rangle = -5, \quad \langle \vec{v}_2, \vec{v}_2 \rangle = 1, \quad \langle \vec{v}_2, \vec{v}_3 \rangle = 0, \quad \langle \vec{v}_3, \vec{v}_3 \rangle = 9.$$

a) Compute $\|\vec{v}_1 + \vec{v}_2\|^2$.

b) Compute $\|\vec{v}_1 + \vec{v}_2 + \vec{v}_3\|^2$.

c) Find the orthogonal projection of $\vec{v}_1$ onto $\mathrm{span}\{\vec{v}_2, \vec{v}_3\}$.

Answer:

a) $\|\vec{v}_1 + \vec{v}_2\|^2 = 10$

b) $\|\vec{v}_1 + \vec{v}_2 + \vec{v}_3\|^2 = 9$

c) $2\vec{v}_2 - \frac{1}{9}\vec{v}_3$