MAT313, Fall 2025

Practice Problems for the Midterm Exam

1. Find all solutions to the system of linear equations

$$\begin{cases} x + 2y - z = 3 \\ 2x + 5y + z = 12 \\ 3x + 7y + 2z = 17 \end{cases}$$

Answer: $(x, y, z) = (-2, 3, 1)$.

2. Find the unique polynomial $p(t) = a_2t^2 + a_1t + a_0$ with $p(1) = 5/2$, $p(2) = 9$, $p(4) = 37$.

Answer: $p(t) = \frac{5}{2}t^2 - t + 1$.

3. Find all solutions to the system of linear equations

$$\begin{cases} x_1 + 4x_2 + 3x_3 + 2x_4 = 0 \\ 8x_1 + 16x_2 + 4x_3 + 4x_4 = 0 \\ 6x_1 + 20x_2 + 5x_3 + 17x_4 = 0 \end{cases}$$

Answer: $x_1 = 3x_4,\ x_2 = -2x_4,\ x_3 = x_4,\ x_4$ free

4. Find the reduced row echelon form of:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}$$

Answer:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

5. Find necessary and sufficient conditions for the system below to be consistent, i.e., to admit solutions. If these conditions hold, how many solutions does the system have?

$$\begin{cases} x_1 + 2x_2 + 3x_3 = u_1 \\ 2x_1 + 4x_2 + 6x_3 = u_2 \\ x_1 + x_2 + x_3 = u_3 \end{cases}$$

Answer: The system is consistent if and only if $2u_1 - u_2 = 0$, in which case it has infinitely many solutions because there is a free variable.

6. Can the vector $v = \begin{bmatrix} 4 \\ 1 \\ 4 \end{bmatrix}$ be written as a linear combination of $w_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $w_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$?

   Answer: Yes: $v = 2w_1 + w_2$

7. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x, y) = (x + y, xy)$. Determine whether T is a linear transformation. If so, then:

   (a) Find the matrix that represents T in the canonical basis.

   (b) Is T injective?

   (c) Is T surjective?

   (d) Is T invertible? If so, find a formula for its inverse.

   Answer: Not linear due to xy term (not additive or scalar multiplicative)

8. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x, y) = (x + y, x - y)$. Determine whether T is a linear transformation. If so, then:

   (a) Find the matrix that represents T in the canonical basis.

   (b) Is T injective?

   (c) Is T surjective?

   (d) Is T invertible? If so, find a formula for its inverse.

   Answer: a) The matrix that represents T in the canonical basis is

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

   b) Yes, it is injective because the columns of the above matrix are linearly independent, so $\ker T = \{0\}$.

   c) Yes, it is surjective because the columns of the above matrix span $\mathbb{R}^2$, so $\operatorname{Im} T = \mathbb{R}^2$.

   d) Yes, it is invertible because it is bijective (both injective and surjective) and T^{-1} is represented in the canonical basis by the matrix $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$.

9. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map, such that $T(1, 2) = (3, 6, 9)$ and $T(2, 7) = (18, -3, 1/2)$.

   (a) Find $T(0, 1)$.

   (b) What is the matrix that represents T in the canonical basis?

   (c) What are the dimensions of the subspaces $\ker T$ and $\operatorname{Im} T$?

Answer: a) $T(0,1) = \left(4, -5, -\frac{35}{6}\right)$,

b) $[T] = \begin{bmatrix} -5 & 4 \\ 16 & -5 \\ \frac{62}{3} & -\frac{35}{6} \end{bmatrix}$

c) $\dim \operatorname{Im} T = 2, \quad \dim \ker T = 0$

10. Determine if the set $\{(1,2,3),(2,4,6),(0,1,1)\}$ is linearly independent.
Answer: No, the first two vectors are linearly dependent.

11. Find the value of t such that the vectors $\begin{bmatrix} t \\ t \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 1+t \\ 2t \\ 5+3t \end{bmatrix}$ are linearly dependent.

Answer: $t = 1$.

12. Write a basis for the subspace of $\mathbb{R}^8$ consisting of solutions to the following system:

$$\begin{cases} x_1 + x_2 = 0 \\ x_2 + x_3 = 0 \\ x_3 + x_4 = 0 \\ x_4 + x_5 = 0 \\ x_5 + x_6 = 0 \\ x_6 + x_7 = 0 \\ x_7 + x_8 = 0 \end{cases}$$

Answer: $\{(-1,1,-1,1,-1,1,-1,1)\}$.

13. Decide if each of the following sets is linearly dependent or linearly independent:

(a) $\{(4,6),(1,1),(3,7)\}$ in $\mathbb{R}^2$
(b) $\{(1,-1),(1,1)\}$ in $\mathbb{R}^2$
(c) $\{(0,1,1),(1,0,1),(1,1,0)\}$ in $\mathbb{R}^3$

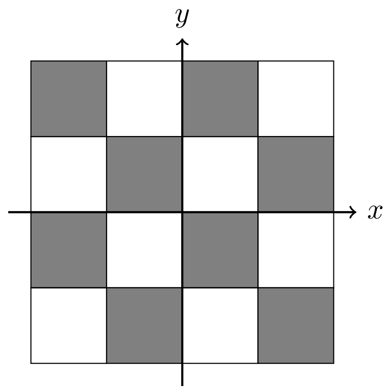
Answer: a) lin dep, b) lin indep, c) lin indep

14. Find the vector obtained by rotating the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ by 45^o counterclockwise.

Answer:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \sqrt{2} - \frac{3}{\sqrt{2}} \\ \sqrt{2} + \frac{3}{\sqrt{2}} \end{bmatrix}$$

15. Sketch the image of the checkerboard below under each of linear transformation:



(a) $T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $T = \begin{bmatrix} 3 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

(c) $T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

16. Is $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ invertible? If so, find its inverse.

Answer: No, since $\det(A) = 0$.

17. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$. Compute $A + B$, $A \cdot B$, $B \cdot A$, A^2, and B^2.

Answer: $A + B = \begin{bmatrix} 3 & 1 \\ 3 & 6 \end{bmatrix}$, $A \cdot B = \begin{bmatrix} 2 & 1 \\ 6 & 2 \end{bmatrix}$, $B \cdot A = \begin{bmatrix} -1 & -1 \\ 3 & 5 \end{bmatrix}$, $A^2 = \begin{bmatrix} 7 & 12 \\ 18 & 31 \end{bmatrix}$, $B^2 = \begin{bmatrix} 4 & -3 \\ 0 & 1 \end{bmatrix}$.

18. Is $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ invertible? If so, find its inverse.

Answer: Yes, since $\det(A) = -1$. The inverse is $A^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$.

19. Find the rank and nullity of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 4 & 5 & 6 \end{bmatrix}$.

Answer: Rank = 2, Nullity = 1

20. Find a basis for $\ker A$, where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \end{bmatrix}$$

Answer: $\left\{ \begin{bmatrix} 1 \\ -1/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

21. Find bases for $\ker T$ and $\operatorname{Im} T$ for each of the following linear transformations:

(a) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (x + y, 2x + 2y)$

(b) $T\colon \mathbb{R}^4 \to \mathbb{R}^2$, $T(x, y, z, w) = (x, w)$

(c) $T\colon \mathbb{R} \to \mathbb{R}^2$, $T(x) = (x, 5x)$

Answer: a) $\ker T = \operatorname{span}\{(1, -1)\}$, $\operatorname{Im} T = \operatorname{span}\{(1, 2)\}$
b) $\ker T = \operatorname{span}\{(0, 1, 0, 0), (0, 0, 1, 0)\}$, $\operatorname{Im} T = \operatorname{span}\{(1, 0), (0, 1)\}$
c) $\ker T = \{0\}$, $\operatorname{Im} T = \operatorname{span}\{(1, 5)\}$

22. Find the dimension of the subspace of $\mathbb{R}^5$ given by vectors of the form

$$\big(a - 3b + c,\ 2a - 6b - 2c,\ 3a - 9b + c,\ c,\ 2a - 6b + 6c\big)$$

Answer: 2