Lecture 10

1. Two-phase simplex method

So far, we have assumed that LPs can be written as

$$\min \quad c^T x \quad \text{s.t.} \quad Ax \leq b, \quad x \geq 0 \tag{1}$$

where $b \geq 0$. Under these assumptions, the LP is always *feasible*, as $x = 0$ is a feasible solution. This also gives us a 'starting tableau' where the 'original' (often called 'decision') variables are set to zero and the 'slack' variables are set equal to the right-hand side values b_j. Let us now discuss the case of a *general LP*, which, up to multiplying certain rows by -1, can be assumed to be of the form

$$\begin{aligned} \min \quad c^T x \quad \text{s.t.} \quad & a_{11}\, x_1 + \cdots + a_{1n}\, x_n \,\square\, b_1, \\ & \cdots \\ & a_{m1}\, x_1 + \cdots + a_{mn}\, x_n \,\square\, b_m, \\ & x \geq 0 \end{aligned} \tag{2}$$

where $b \geq 0$ and $\square$ stands for either $\leq$, $=$, or $\geq$. Recall that a LP of the above form need not be feasible!

The goal is to build on the simplex method we discussed earlier for LPs of the form (1) to have a similar algorithm that handles any input LP of the form (2) and, after finitely many steps, outputs one of the following options:

- LP is infeasible;
- LP is unbounded;
- LP is feasible and bounded, and optimal solution is x_{opt}.

The simplex method algorithm accomplishing this is the *two-phase simplex method.* The first phase is to solve an auxiliary LP to determine feasibility of the original LP and obtain (if possible) an initial basic feasible solution; the second phase is similar to our earlier discussion, either finding an optimal solution or determining that the original LP is unbounded.

1.1. Getting ready.

Given (2), where $b \geq 0$, we proceed as follows:

1. Let $V = \{i_1, \ldots, i_k\}$ be the list of rows $a_{i1}x_1 + \cdots + a_{in}x_n \,\square\, b_i$ where $\square$ is $=$ or $\geq$.
2. Add a slack variable to rows where $\square$ is $\leq$ and *subtract* a slack variable[1] to rows where $\square$ is $\geq$.
3. If $i \in V$, then add an *artificial variable* to row i.

All new variables introduced above are assumed nonnegative. For example, given

$$\begin{aligned} \max \quad 4x_1 + 5x_2 \quad \text{s.t.} \quad & 2x_1 + 3x_2 \leq 6, \\ & 3x_1 + x_2 \geq 3, \\ & x \geq 0, \end{aligned}$$

the steps above are

[1] These are usually called *excess* or *surplus* variables.

(1) $V = \{2\}$.
(2) Add a slack variable x_3 to row 1, subtract a slack variable x_4 from row 2.
(3) Add an artificial variable x_5 to row 2.

Altogether, we obtain the following LP in $x = (x_1, \ldots, x_5)$

$$\begin{aligned} \max \quad 4x_1 + 5x_2 \quad \text{s.t.} \quad & 2x_1 + 3x_2 + x_3 = 6, \\ & 3x_1 + x_2 - x_4 + x_5 = 3, \\ & x \geq 0. \end{aligned} \tag{3}$$

1.2. **Phase I.** In the first phase, we shall determine if the LP is feasible, and, if so, compute a basic feasible solution. This is based on the following elementary result:

Proposition 1. *Let A be an $m \times n$ matrix. The LP on $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ given by*

$$\max\ c^T x \quad \text{s.t.} \quad Ax = b,\ x \geq 0,$$

is feasible if and only if the LP on $\bar{x} = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) \in \mathbb{R}^{n+m}$ given by

$$\min\ x_{n+1} + \cdots + x_{n+m} \quad \text{s.t.} \quad (A \,|\, \mathrm{Id}_m)\bar{x} = b,\ \bar{x} \geq 0,$$

has optimal value 0, where $(A \,|\, \mathrm{Id}_m)$ is the $m \times (n+m)$ matrix obtained juxtaposing A and the $m \times m$ identity matrix.

Exercise 1. Prove Proposition 1.

We shall apply Proposition 1 to our setup with $x_{n+1}, \ldots, x_{n+m}$ being the union of all artificial variables (which we want to get rid of) and slack variables added to rows where $\square$ is $\leq$. Note this gives us an 'obvious' basic feasible solution, as the columns in the tableau corresponding to $x_{n+1}, \ldots, x_{n+m}$ form an $m \times m$ identity matrix. Since we want to get rid of artificial variables, the target function to be minimized in the auxiliary LP is their sum.

For example, the auxiliary LP to determine feasibility of (3) is

$$\begin{aligned} \min \quad x_5 \quad \text{s.t.} \quad & 2x_1 + 3x_2 + x_3 = 6, \\ & 3x_1 + x_2 - x_4 + x_5 = 3, \\ & x \geq 0. \end{aligned}$$

This is arranged so that $x = (0, 0, 6, 0, 3)$ is an obvious basic feasible solution, with tableau

	x_1	x_2	x_3	x_4	x_5	
x_3	2	3	1	0	0	6
x_5	3	1	0	−1	1	3
	0	0	0	0	−1	0

Note that the auxiliary target function takes value 3 at this basic feasible solution. However, there are nonzero entries in the target row on the columns of x_3 and x_5. In order to rectify this, we perform row operations (in this case, just adding the row of x_5 to the target row) and obtain an equivalent tableau satisfying the usual properties:

	x_1	x_2	x_3	x_4	x_5	
x_3	2	3	1	0	0	6
x_5	3	1	0	-1	1	3
	3	1	0	-1	0	3

We now proceed with the simplex method to find the optimal (minimum) value. Using entering variable x_1 we compute $\theta(x_3) = 3$, $\theta(x_5) = 1$ so the departing variable is x_5, and we arrive to

	x_1	x_2	x_3	x_4	x_5	
x_3	0	$\frac{7}{3}$	1	$\frac{2}{3}$	$-\frac{2}{3}$	4
x_1	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{1}{3}$	1
	0	0	0	0	-1	0

By the above, the minimum value of the auxiliary target function is 0 hence, by Proposition 1, the LP (3) is feasible with artificial variables set to 0. Namely, $x = (1, 0, 4, 0, 0)$ is a basic feasible solution corresponding to the feasible basis $B = \{1, 3\} \subset \{1, \ldots, 5\}$.

Exercise 2. Check that $x = (1, 0, 4, 0)$ is a basic feasible solution for (3).

We then remove the artificial variable x_5 and proceed to Phase II. If the optimal (minimum) value at the end of Phase I is > 0, then the original LP is not feasible and the algorithm terminates.

1.3. **Phase II.** Using the basic feasible solution resulting from Phase I and the original target function, we build the tableau

	x_1	x_2	x_3	x_4	
x_3	0	$\frac{7}{3}$	1	$\frac{2}{3}$	4
x_1	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	1
	-4	-5	0	0	0

Again, note the above does not yet satisfy the usual tableau properties. Performing row operations to eliminate the nonzero entries in the target row for columns of basic variables, we obtain:

	x_1	x_2	x_3	x_4	
x_3	0	$\frac{7}{3}$	1	$\frac{2}{3}$	4
x_1	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	1
	0	$-\frac{11}{3}$	0	$-\frac{4}{3}$	4

Exercise 3. Finish the example above to find that the maximum value of the target function is 12, which is attained at the basic feasible solution $x = (1, 0, 0, 6)$ with $B = \{1, 4\}$.

Solution to Exercise 3. See `lecture10.nb`.