

Lecture 16

1. REVIEW OF LINEAR ALGEBRA

Recall that $\lambda \in \mathbb{R}$ is an *eigenvalue* for an $n \times n$ matrix A if the linear equation $Ax = \lambda x$ has a nontrivial solution $x \neq 0$. In other words, the kernel (or nullspace) of $\lambda \text{Id} - A$ is nontrivial; so, equivalently, the eigenvalues of A are the roots of the *characteristic polynomial* $p(\lambda) = \det(\lambda \text{Id} - A)$, which is a monic polynomial of degree n in λ .

If λ is an eigenvalue of A , the elements of $\ker(\lambda \text{Id} - A)$ are called the *eigenvectors* of A associated to the eigenvalue λ , and the linear subspace $\ker(\lambda \text{Id} - A)$ is called the *eigenspace* of A associated to the eigenvalue λ .

Exercise 1. Find all eigenvalues of the following matrices; for each eigenvalue, find a basis of the corresponding eigenspace.

a) $A = \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}$

b) $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 4 \end{pmatrix}$

c) $A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

The collection of all eigenvalues of the matrix A is called its *spectrum*, and we denote it by $\text{Spec}(A) = \{\lambda_i \in \mathbb{R} : \lambda_i \text{ is an eigenvalue of } A\}$. The matrix A is *diagonalizable* if there is a decomposition $\mathbb{R}^n = \bigoplus_{\lambda_i \in \text{Spec}(A)} \ker(\lambda_i \text{Id} - A)$ into eigenspaces associated to eigenvalues of A . Equivalently, A is diagonalizable if there exists an invertible matrix P such that $A = PDP^{-1}$, where $D = \text{diag}(\lambda_i)$ is a diagonal matrix (in this case, the columns of P are the coordinates of eigenvectors of A). Which of the matrices above are diagonalizable, and with which P ?

Note that the behavior of A as a map $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is geometrically clear in each eigenspace (it is simply multiplication by λ_i) so if A is diagonalizable, then we can find out what A does to any vector in \mathbb{R}^n by decomposing it into the components in each eigenspace of A . In other words, if we ‘change basis’ using P and work on a basis of eigenvectors of A , then A operates as the diagonal matrix $D = \text{diag}(\lambda_i)$. This is illustrated in the diagram below:

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{A} & \mathbb{R}^n \\ \uparrow P & & \uparrow P \\ \mathbb{R}^n & \xrightarrow{D} & \mathbb{R}^n \end{array}$$

We will make frequent use of the following important algebraic results.

Theorem 1. *If A is symmetric, that is, $A^T = A$, then A is orthogonally diagonalizable, that is, there exists a matrix P such that $P^T P = \text{Id}$ and $A = PDP^{-1} = PDP^T$, where D is diagonal.*

Theorem 2. *The characteristic polynomial $p(\lambda)$ of an $n \times n$ matrix A satisfies*

$$p(\lambda) = \det(\lambda \text{Id} - A) = \sum_{k=0}^n (-1)^k \text{tr}(\wedge^k A) \lambda^{n-k},$$

where $\text{tr}(\wedge^k A)$ is the sum of all principal minors¹ of A of size k .

¹Recall that a principal minor of A of size k is the determinant of a submatrix of A obtained by selecting rows $\{i_1, \dots, i_k\}$ and columns $\{i_1, \dots, i_k\}$.