Lecture 19/20

1. Semidefinite programming

A *semidefinite program* (SDP) is an optimization problem of the form

$$\min \quad \langle C, X\rangle \quad \text{s.t.} \quad \langle A_i, X\rangle = b_i, \qquad i = 1,\dots,m, \qquad X \succeq 0 \tag{1}$$

where $A_i, C \in \mathrm{Sym}^2(\mathbb{R}^n)$ are given $n\times n$ symmetric matrices, and $b_i \in \mathbb{R}$. Recall that the inner product in the vector space $\mathrm{Sym}^2(\mathbb{R}^n)$ of symmetric $n\times n$ matrices is given by $\langle X, Y\rangle = \operatorname{tr} XY$.

In other words, an SDP is an optimization problem whose target function is *linear* and whose feasible set is a *spectrahedron*.

Exercise 1. [1] Recognize that the following problem is an SDP on the matrix $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22}\end{pmatrix}$.

$$\min \quad 2x_{11} + 2x_{12} \quad \text{s.t.} \quad x_{11} + x_{22} = 1, \quad X \succeq 0.$$

Find the optimal solution by recognizing (geometrically) the feasible set as a subset of $\mathbb{R}^2$.

Solution to Exercise 1. The above is an SDP of the form (1) with

$$C = \begin{pmatrix} 2 & 1 \\ 1 & 0\end{pmatrix},\ A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix},\ b_1 = 1.$$

The feasible set is the spectrahedron determined by the polynomial inequality $x_{11}(1-x_{11}) \geq x_{12}^2$, which can be easily seen (complete the square!)[2] to be a closed disk in the $(x_{11}, x_{12}, 1-x_{11})$-plane with center $(1/2, 0, 1/2)$ and radius $1/2$. Thus, the optimal solution is

$$X = \begin{pmatrix} \frac{2-\sqrt{2}}{4} & -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{2+\sqrt{2}}{4}\end{pmatrix},$$

where the target function achieves its minimum value $1-\sqrt{2}$.

2. Spectrahedral shadows

A subset $S \subset \mathbb{R}^n$ is a *spectrahedral shadow* if there exists a spectrahedron $S' \subset \mathbb{R}^m$ and an affine-linear map $f\colon \mathbb{R}^N \to \mathbb{R}^n$ such that $S = f(S')$. In other words, it is a set of the form

$$S = \left\{ x \in \mathbb{R}^n : \exists y \in \mathbb{R}^m,\ A_0 + \sum_{i=1}^n x_i A_i + \sum_{j=1}^m y_j B_j \succeq 0 \right\},$$

where $A_i, B_j \in \mathrm{Sym}^2(\mathbb{R}^d)$ are symmetric matrices. Note that, letting $f\colon \mathbb{R}^n \oplus \mathbb{R}^m \to \mathbb{R}^n$ be the projection map $f(x,y) = x$, we have that $S = f(S')$, where S' is the spectrahedron

$$S' = \left\{ (x,y) \in \mathbb{R}^n \oplus \mathbb{R}^m : A_0 + \sum_{i=1}^n x_i A_i + \sum_{j=1}^m y_j B_j \succeq 0 \right\}.$$

Optimization problems where the target function is linear and the feasible region is a spectrahedral shadow can be solved as a semidefinite program (SDP) by introducing slack variables.

[1] This exercise is taken from "Semidefinite Optimization and Convex Algebraic Geometry", MOS-SIAM Series on Optimization, edited by G. Blekherman, P. Parrilo, and R. Thomas.

[2] To find that this is equivalent to $\left(x_{11} - \frac{1}{2}\right)^2 + x_{12}^2 \leq \frac{1}{4}$.

Exercise 2. [3] Describe geometrically the spectrahedron

$$S' = \left\{ (x,y,z) \in \mathbb{R}^3 : \begin{pmatrix} z+y & 2z-x & 0 \\ 2z-x & z-y & 0 \\ 0 & 0 & 1-z \end{pmatrix} \succeq 0 \right\},$$

and the spectrahedral shadow $f(S')$, where $f\colon \mathbb{R}^3 \to \mathbb{R}$ is given by $f(x,y,z) = (x,y)$.

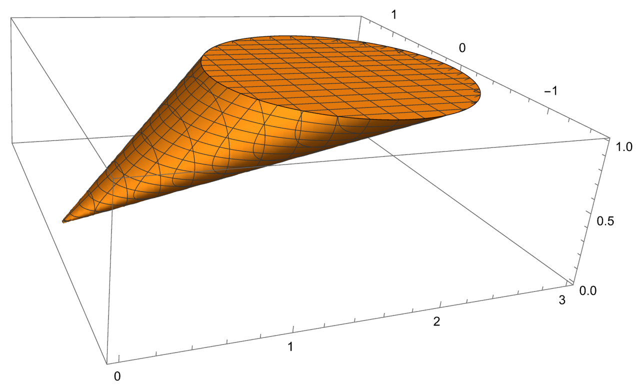
Solution to Exercise 2. As the matrix is block-diagonal, it is positive-semidefinite if and only if

$$z \pm y \geq 0 \quad \text{and} \quad z^2 - y^2 - (2z-x)^2 \geq 0 \quad \text{and} \quad z \leq 1,$$

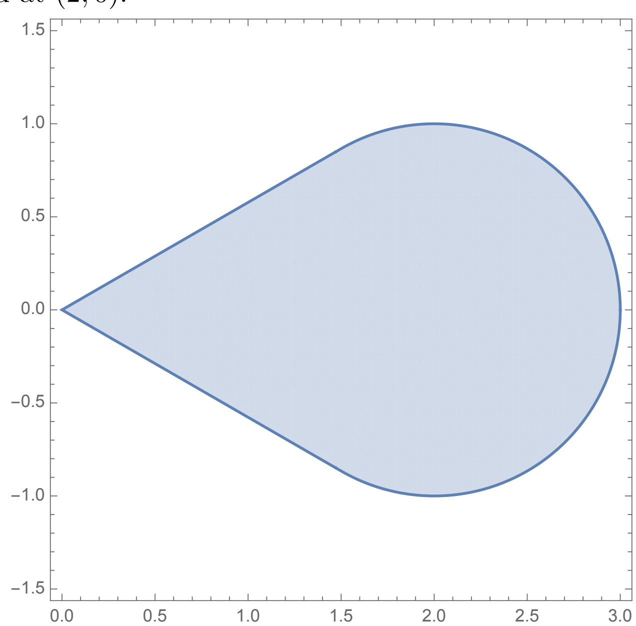
which is equivalent to

$$|y| \leq z \leq 1 \quad \text{and} \quad (x-2z)^2 + y^2 \leq z^2.$$

For each $0 \leq z_0 \leq 1$, the above describes a closed disk in the plane (x,y,z_0) with center $(2z_0, 0, z_0)$ and radius z_0. Thus, the spectrahedron $S' \subset \mathbb{R}^3$ is the cone in $\mathbb{R}^3$ given by the convex hull of the origin $(0,0,0)$ and the disk of radius 1 in the plane $(x,y,1)$ centered at $(2,0,1)$.



Accordingly, the spectrahedral shadow $f(S') \subset \mathbb{R}^2$ is the convex hull of the origin $(0,0)$ and the disk of radius 1 centered at $(2,0)$.



[3]This exercise is taken from "Semidefinite Optimization and Convex Algebraic Geometry", MOS-SIAM Series on Optimization, edited by G. Blekherman, P. Parrilo, and R. Thomas.

Exercise 3. a) Use Mathematica to plot the *old-fashioned TV screen* given by

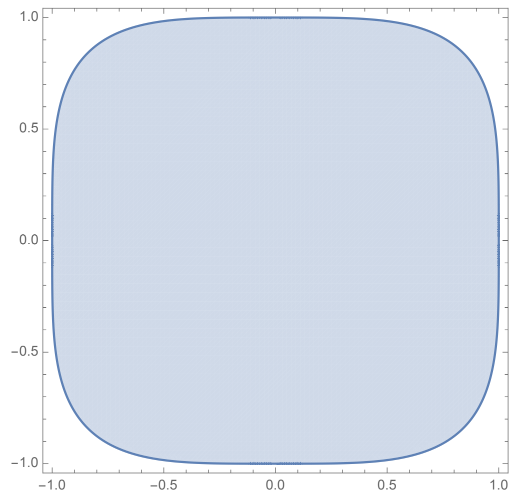
$$C = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^4 + x_2^4 \leq 1\right\}.$$

b) Compute (geometrically) the maximum value of $x_1 + x_2$ among $(x_1, x_2) \in C$.
c) Check that C can be written as a spectrahedral shadow

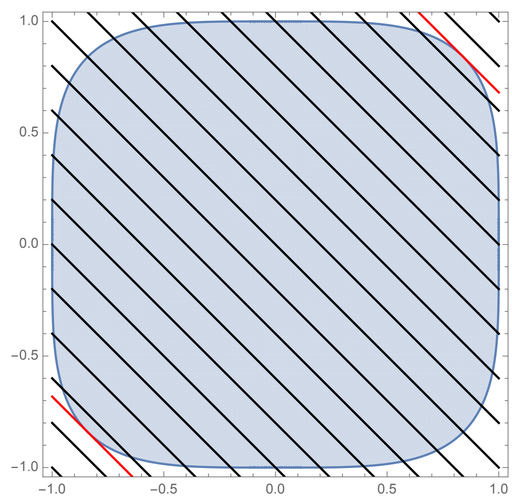
$$S = \left\{x \in \mathbb{R}^2 : \exists y \in \mathbb{R}^2, \begin{pmatrix} 1 & x_1 \\ x_1 & y_1 \end{pmatrix} \succeq 0, \begin{pmatrix} 1 & x_2 \\ x_2 & y_2 \end{pmatrix} \succeq 0, \begin{pmatrix} 1 - y_1 & y_2 \\ y_2 & 1 + y_1 \end{pmatrix} \succeq 0\right\},$$

and use this to write an SDP that confirms the computation in b).
d) Taking inspiration from the above, can you represent the set $\left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^6 + x_2^6 \leq 1\right\}$ as a spectrahedral shadow? What about $\left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^{2k_1} + x_2^{2k_2} \leq 1\right\}$ for any given $k_1, k_2 \in \mathbb{N}$?

Solution to Exercise 3. a) The plot of the region C is as follows:



b) Geometrically, we find that the maximum value is $\frac{2}{\sqrt[4]{2}}$, which is achieved when $x_1 = x_2 = \frac{1}{\sqrt[4]{2}}$.



c) Clearly, $x_1^4 + x_2^4 \leq 1$ if and only if $(x_1^2)^2 + (x_2^2)^2 \leq 1$. If $x \in C$, setting $y = (x_1^2, x_2^2)$ we see that $x \in S$. Recall that positive-semidefiniteness of the third 2×2 matrix in the definition of S describes a disk of radius 1 centered at the origin in the (y_1, y_2)-plane. Thus, conversely, if $x \in S$, then $x_i^2 \le y_i$ and $y_1^2 + y_2^2 \le 1$, hence $x_1^4 + x_2^4 \le 1$, which means $x \in C$. The SDP is simply:

$$\max\ x_1 + x_2 \quad \text{s.t.} \quad \begin{pmatrix} 1 & x_1 & & & & \\ x_1 & y_1 & & & & \\ & & 1 & x_2 & & \\ & & x_2 & y_2 & & \\ & & & & 1 - y_1 & y_2 \\ & & & & y_2 & 1 + y_1 \end{pmatrix} \succeq 0.$$

d) To obtain x_1 to the power $2k_1$, $k_1 \in \mathbb{N}$, set $z_{1,1} = x_1$ and use k_1 auxiliary variables $z_{1,2}, \dots, z_{1,k_1+1}$ such that $\begin{pmatrix} 1 & z_{1,j} \\ z_{1,j} & z_{1,j+1} \end{pmatrix} \succeq 0$, for $j = 1, \dots, k_1$, which is equivalent to $x_1^{2k_1} \le z_{1,k_1+1}$. Proceed analogously to obtain $x_2^{2k_2} \le z_{2,k_2+1}$. Finally, use the spectrahedral description of the unit disk on $(z_{1,k_1+1}, z_{2,k_2+1})$, as above.