

## Lecture 23

## 1. NONNEGATIVE V. SUMS OF SQUARES

We say that a polynomial  $p \in \mathbb{R}[x]_{2d}$  in  $n$  variables  $x_1, \dots, x_n$  of degree  $\leq 2d$  is

- i) *nonnegative* if  $p(x_1, \dots, x_n) \geq 0$  for all  $x \in \mathbb{R}^n$ ;
- ii) a *sum of squares (sos)* if there exist polynomials  $q_j \in \mathbb{R}[x]_d$  such that  $p(x) = \sum_{j=1}^k q_j(x)^2$ .

Clearly, every sos polynomial is nonnegative. Positive multiples and convex combinations of nonnegative polynomials (respectively, sos polynomials) are again nonnegative (respectively, sos), thus

$$P_{n,2d} := \{p \in \mathbb{R}[x]_{2d} : p \text{ is nonnegative}\}$$

$$\Sigma_{n,2d} := \{p \in \mathbb{R}[x]_{2d} : p \text{ is sos}\}$$

are convex cones in the vector space  $\mathbb{R}[x]_{2d} \cong \mathbb{R}^N$ , where  $N = \binom{n+d}{d}$ .

**Exercise 1.** Prove that  $P_{n,2d}$  is a closed semialgebraic set. Find an explicit description of  $P_{1,2}$  as a semialgebraic set.

**Solution to Exercise 1.** This is a consequence of Quantifier Elimination, by eliminating the quantifier  $\forall x$  in  $\forall x, p(x) \geq 0$ . For example, for univariate quadratic polynomials, we have that

$$\begin{aligned} P_{1,2} &= \{p(x) = ax^2 + bx + c : \forall x, p(x) \geq 0\} \\ &= \{p(x) = ax^2 + bx + c : \min_{x \in \mathbb{R}} p(x) \geq 0\} \\ &= \{p(x) = ax^2 + bx + c : a > 0 \text{ and } p(-b/2a), \text{ or } a = b = 0 \text{ and } p(0) \geq 0\} \\ &= \{ax^2 + bx + c : a > 0 \text{ and } b^2 - 4ac \leq 0, \text{ or } a = b = 0 \text{ and } c \geq 0\} \\ &= \{ax^2 + bx + c : a \geq 0, c \geq 0, 4ac - b^2 \geq 0\}. \end{aligned}$$

Note that  $P_{1,2}$  is actually *basic* semialgebraic, and it is also a spectrahedron:

$$\begin{aligned} P_{1,2} &= \left\{ ax^2 + bx + c : \forall x, \begin{pmatrix} 1 \\ x \end{pmatrix}^T \begin{pmatrix} c & b/2 \\ b/2 & a \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} \geq 0 \right\} \\ &= \left\{ ax^2 + bx + c : \begin{pmatrix} c & b/2 \\ b/2 & a \end{pmatrix} \succeq 0 \right\} \end{aligned}$$

As it turns out,  $P_{n,2d}$  is semialgebraic but it is not *basic* semialgebraic if  $2d \geq 4$ .<sup>1</sup> In particular,  $P_{n,2d}$  is not a spectrahedron if  $2d \geq 4$ .

Since  $\Sigma_{n,2d} \subset P_{n,2d}$ , a natural question is whether the converse inclusion  $\Sigma_{n,2d} \supset P_{n,2d}$  holds, i.e., if nonnegative polynomials are sos. Remarkably, this is almost never the case:

**Theorem 1** (Hilbert, 1888). *The only cases in which  $\Sigma_{n,2d} = P_{n,2d}$  are:*

- a) *Univariate polynomials, i.e.,  $n = 1$ ;*
- b) *Quadratic polynomials, i.e.,  $2d = 2$ ;*
- c) *Bivariate quartics, i.e.,  $n = 2, 2d = 4$ .*

An example of  $p \in P_{2,6} \setminus \Sigma_{2,6}$  is the Motzkin polynomial  $p(x, y) = x^4y^2 + x^2y^4 - 3x^2y^2 + 1$ .

**Exercise 2.** Use the arithmetic-geometric inequality applied to  $\{x^4y^2, x^2y^4, 1\}$  to show that the Motzkin polynomial is nonnegative. Think about how you would try to show it is not sos.

<sup>1</sup>For a more details, see p. 52 in [BPT13].

**Solution to Exercise 2.** By the arithmetic-geometric inequality, we have that

$$\frac{x^4y^2 + x^2y^4 + 1}{3} \geq \sqrt[3]{(x^4y^2)(x^2y^4)} = x^2y^2,$$

that is,  $p(x, y) \geq 0$ . If  $p(x, y)$  was sos, then one can show it would be a sum of terms of the form  $(ax^2y + bxy^2 + cxy + d)^2$  but no such term has a negative coefficient for  $x^2y^2$ .

**Exercise 3.** Use the computations in Exercise 1 to show that  $\Sigma_{1,2} = P_{1,2}$ . Think about how this could be generalized prove the equality in Theorem 1 b).

**Solution to Exercise 3.** Let us use the spectrahedral description

$$P_{1,2} = \left\{ ax^2 + bx + c : M(a, b, c) := \begin{pmatrix} c & b/2 \\ b/2 & a \end{pmatrix} \succeq 0 \right\}.$$

Since  $M(a, b, c) \succeq 0$ , there exists a matrix  $P$  such that  $M(a, b, c) = P^T P$ . Using, e.g., the Cholesky decomposition, we find that, with  $a \geq 0$ ,  $c \geq 0$ ,  $4ac - b^2 \geq 0$ , assuming for simplicity  $c > 0$ ,<sup>2</sup>

$$P = \begin{pmatrix} \sqrt{c} & \frac{b}{2\sqrt{c}} \\ 0 & \frac{\sqrt{4ac-b^2}}{2\sqrt{c}} \end{pmatrix}.$$

Thus, setting  $v = \begin{pmatrix} 1 \\ x \end{pmatrix}$ , we have

$$ax^2 + bx + c = v^T M(a, b, c) v = v^T (P^T P) v = (Pv)^T (Pv).$$

Since  $Pv = \left( \frac{bx}{2\sqrt{c}} + \sqrt{c}, \frac{x\sqrt{4ac-b^2}}{2\sqrt{c}} \right)$ , the above yields the following sos decomposition:

$$ax^2 + bx + c = \left( \frac{bx}{2\sqrt{c}} + \sqrt{c} \right)^2 + \left( \frac{x\sqrt{4ac-b^2}}{2\sqrt{c}} \right)^2.$$

Therefore,  $P_{1,2} \subset \Sigma_{1,2}$  and hence  $P_{1,2} = \Sigma_{1,2}$ .

One can prove the equality in Theorem 1 b) using a similar reasoning, identifying quadratic polynomials in  $P_{n,2}$  with  $(n+1) \times (n+1)$  positive-semidefinite matrices operating on  $v = (1, x_1, x_2, \dots, x_n)$ . In particular,  $\Sigma_{n,2} = P_{n,2}$  is a spectrahedron.

More generally, for general  $d$ , it can be shown that  $\Sigma_{n,2d}$  is a spectrahedral shadow<sup>3</sup> of dimension  $\binom{n+2d}{2d}$ , see [BPT13, Cor 3.40]. In particular, SDP can be used to test if  $p \in \Sigma_{n,2d}$  and find polynomials  $q_j \in \mathbb{R}[x]_d$  such that  $p = \sum_j q_j^2$ . Let us illustrate this with an example of  $p \in \Sigma_{1,4}$ .

**Exercise 4.** ([BPT13, Ex. 3.35]) Use the following steps to find an sos decomposition for the nonnegative polynomial  $p(x) = x^4 + 4x^3 + 6x^2 + 4x + 5$ .

- i) Let  $v = (1, x, x^2)$  be the vector consisting of monomials of degree  $\leq 2$  and  $M \in \text{Sym}^2(\mathbb{R}^3)$  be a symmetric matrix. Determine affine-linear constraints on  $M$  equivalent to  $p(x) = v^T M v$ , by matching coefficients.
- ii) Use SDP to find  $M \succeq 0$  satisfying the above constraints.
- iii) Find  $P$  such that  $M = P^T P$  and show that  $Pv$  yields the desired sos decomposition.

**Solution to Exercise 4.** Using basic calculus, we see that  $p(x) \geq 4$  for all  $x \in \mathbb{R}$ , thus we have that  $p \in P_{1,4} = \Sigma_{1,4}$ , hence an sos decomposition exists.

<sup>2</sup>How would you handle the case  $c = 0$ ? What are the possible values of  $a, b$ ?

<sup>3</sup>But  $\Sigma_{n,2d}$  is not a spectrahedron, unless  $2d = 2$ .

i) Writing  $M = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{pmatrix}$ , we have that

$$v^T M v = a_{00} + 2a_{01}x + x^2(2a_{02} + a_{11}) + 2a_{12}x^3 + a_{22}x^4.$$

So, matching coefficients with  $p(x) = x^4 + 4x^3 + 6x^2 + 4x + 5$  we find the linear constraints

$$a_{00} = 5, \quad a_{01} = 2, \quad a_{02} = 3 - a_{11}/2, \quad a_{12} = 2, \quad a_{22} = 1,$$

and  $a_{11}$  is free.

ii) Substituting these affine-linear constraints in  $M$ , we have

$$M = \begin{pmatrix} 5 & 2 & 3 - \frac{a_{11}}{2} \\ 2 & a_{11} & 2 \\ 3 - \frac{a_{11}}{2} & 2 & 1 \end{pmatrix}.$$

By Sylvester's criterion, it is easy to see that the above is positive-semidefinite if and only if  $4 \leq a_{11} \leq 8$ .

iii) Take, e.g.,  $a_{11} = 6$ , so that  $M = \begin{pmatrix} 5 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 1 \end{pmatrix} \succ 0$ , and use the Cholesky decomposition to find

$P$  such that  $M = P^T P$ . Namely, we have

$$P = \begin{pmatrix} \sqrt{5} & \frac{2}{\sqrt{5}} & 0 \\ 0 & \sqrt{\frac{26}{5}} & \sqrt{\frac{10}{13}} \\ 0 & 0 & \sqrt{\frac{3}{13}} \end{pmatrix},$$

hence  $p(x) = v^T M v = v^T (P^T P) v = (Pv)^T Pv$  yields the sos decomposition

$$p(x) = (Pv)^T Pv = \left( \frac{2x}{\sqrt{5}} + \sqrt{5} \right)^2 + \left( \sqrt{\frac{10}{13}}x^2 + \sqrt{\frac{26}{5}}x \right)^2 + \left( \sqrt{\frac{3}{13}}x^2 \right)^2.$$

As usual, this decomposition is not unique. For instance, the matrix

$$Q = \begin{pmatrix} 0 & 2 & 1 \\ \sqrt{2} & \sqrt{2} & 0 \\ \sqrt{3} & 0 & 0 \end{pmatrix}$$

also satisfies  $M = Q^T Q$  and hence yields the sos decomposition

$$p(x) = (Qv)^T Qv = (x^2 + 2x)^2 + (\sqrt{2} + \sqrt{2}x)^2 + (\sqrt{3})^2.$$

Can you find an sos decomposition with only 2 squares?<sup>4</sup>

#### REFERENCES

- [BPT13] G. BLEKHERMAN, P. A. PARRILO, AND R. R. THOMAS. *Semidefinite optimization and convex algebraic geometry*, vol. 13 of MOS-SIAM Series on Optimization. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA; Mathematical Optimization Society, Philadelphia, PA, 2013.

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<sup>4</sup>For nonnegative univariate polynomials, it is always possible to find an sos decomposition with only 2 squares!