Lecture 24

1. Duality

1.1. **LP.** Recall from Lectures 14-15 that the dual of the primal LP

$$\min \quad c^T x \quad \text{s.t.} \quad \begin{aligned} Ax &= b, \\ x &\geq 0, \end{aligned}$$

is given by

$$\max \quad b^T y \quad \text{s.t.} \quad A^T y \leq c.$$

Feasible solutions give upper/lower bounds for the optimal value of the dual problem, since

$$c^T x - b^T y = x^T c - (Ax)^T y = x^T (c - A^T y) \geq 0.$$

Moreover, the Strong Duality Theorem ensured that if both primal and dual are feasible, then optimal solutions x_* and y_* exist and the corresponding optimal values agree, that is, $c^T x_* = b^T y_*$.

1.2. **SDP.** Given the primal SDP on the variable $X \in \mathrm{Sym}^2(\mathbb{R}^d)$,

$$\min \quad \langle C, X \rangle \quad \text{s.t.} \quad \begin{aligned} &\langle A_i, X \rangle = b_i, \quad i = 1, \dots, m, \\ &X \succeq 0, \end{aligned}$$

the dual SDP is

$$\max \quad b^T y \quad \text{s.t.} \quad \sum_{i=1}^m y_i A_i \preceq C,$$

on the variable $y \in \mathbb{R}^m$. Similarly to LPs, feasible solutions give upper/lower bounds for the dual:

$$\langle C, X \rangle - b^T y = \langle C, X \rangle - \sum_{i=1}^m y_i \langle A_i, X \rangle = \left\langle C - \sum_{i=1}^m y_i A_i, X \right\rangle \geq 0.$$

Note that the feasible sets of the primal and dual SDP are spectrahedra described in 'different' ways; see Exercise 3 in Lecture 18.

Exercise 1. Justify the last inequality: prove that if $P, Q \in \mathrm{Sym}^2(\mathbb{R}^d)$ satisfy $P \succeq 0$ and $Q \succeq 0$, then $\langle P, Q \rangle \geq 0$ and equality holds if and only if $PQ = QP = 0$. Hint: $P = R^T R$, $Q = S^T S$.

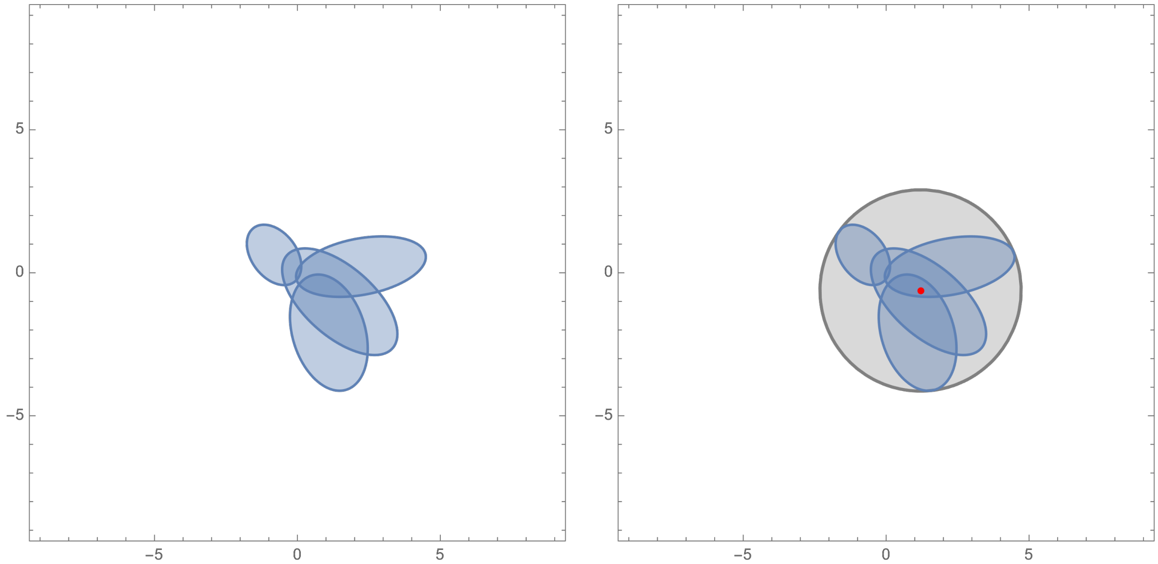
Solution to Exercise 1. Following the hint, recall that since $P, Q \in \mathrm{Sym}^2(\mathbb{R}^d)$ satisfy $P \succeq 0$ and $Q \succeq 0$, there exist $R, S \in \mathrm{Sym}^2(\mathbb{R}^d)$ such that $P = R^T R$, $Q = S^T S$. Then,

$$\begin{aligned} \langle P, Q \rangle = \operatorname{tr} PQ^T = \operatorname{tr}(R^T R)(S^T S)^T &= \operatorname{tr} R^T R S^T S \\ &= \operatorname{tr}(RS^T)(SR^T) = \operatorname{tr}(RS^T)(RS^T)^T = \langle RS^T, RS^T \rangle \geq 0. \end{aligned}$$

However, *strong* duality fails for SDP, see [BPT13, Ex 2.14] for an example. In order to have equality between the optimal values of primal and dual SDP, a sufficient condition is that both are *strictly feasible*, i.e., there exists $X \succ 0$ satisfying the constraints of the primal and there exists y such that $C - \sum_i y_i A_i \succ 0$ for the dual.

2. Using SDP to solve a geometric problem

Consider the problem of finding the smallest disk in $\mathbb{R}^2$ that contains a given number of ellipses.[1] Besides its computational geometry appeal, this problem arises naturally in various computer vision settings, e.g., in connection with object detection, bounding volumes, collision avoidance, etc.



Assume the ellipses are the sublevelsets $\mathcal{E}_i = \{x \in \mathbb{R}^2 : q_i(x) \leq 0\}$ of the quadratic functions

$$q_i(x) = x^T A_i x + 2b_i^T x + c_i,$$

where $A_i \in \mathrm{Sym}^2(\mathbb{R}^2)$ is a positive-semidefinite matrix, $b_i \in \mathbb{R}^2$, $c_i \in \mathbb{R}$. We shall use the following:

Proposition 1. *The ellipse $\mathcal{E} = \{x \in \mathbb{R}^2 : q(x) \leq 0\}$ contains the ellipse $\overline{\mathcal{E}} = \{x \in \mathbb{R}^2 : \overline{q}(x) \leq 0\}$, where $q(x) = x^T A x + 2b^T x + c$ and $\overline{q}(x) = x^T \overline{A} x + 2\overline{b}^T x + \overline{c}$ if and only if there is $\tau \geq 0$ such that*[2]

$$\begin{pmatrix} A & b \\ b^T & c \end{pmatrix} \preceq \tau \begin{pmatrix} \overline{A} & \overline{b} \\ \overline{b}^T & \overline{c} \end{pmatrix}.$$

Thus, a circle $\mathcal{C} = \{x \in \mathbb{R}^2 : q_c(x) \leq 0\}$, where $q_c(x) = x^T x - 2x_c^T x + \gamma$ contains the ellipses $\mathcal{E}_i$, $i = 1, \ldots, p$, if and only if there exists $\tau_i \geq 0$ such that

$$\begin{pmatrix} \mathrm{Id} & -x_c \\ -x_c^T & \gamma \end{pmatrix} \preceq \tau_i \begin{pmatrix} A_i & b_i \\ b_i^T & c_i \end{pmatrix}, \quad i = 1, \ldots, p.$$

Exercise 2. a) Show that the radius of the circle $\mathcal{C}$ is $\sqrt{x_c^T x_c - \gamma}$.
b) Write an SDP that is equivalent to the geometric optimization problem at hand:

$$\min \quad \sqrt{x_c^T x_c - \gamma} \quad \text{s.t.} \quad \begin{pmatrix} \mathrm{Id} & -x_c \\ -x_c^T & \gamma \end{pmatrix} \preceq \tau_i \begin{pmatrix} A_i & b_i \\ b_i^T & c_i \end{pmatrix}, \quad i = 1, \ldots, p.$$
$$\tau_i \geq 0, \quad i = 1, \ldots, p.$$

Solution to Exercise 2. a) The radius can be found by completing the square in $q_c(x) \leq 0$.
b) In order to minimize the radius of $\mathcal{C}$, we minimize t such that

$$\begin{pmatrix} \mathrm{Id} & x_c \\ x_c^T & t + \gamma \end{pmatrix} \succeq 0.$$

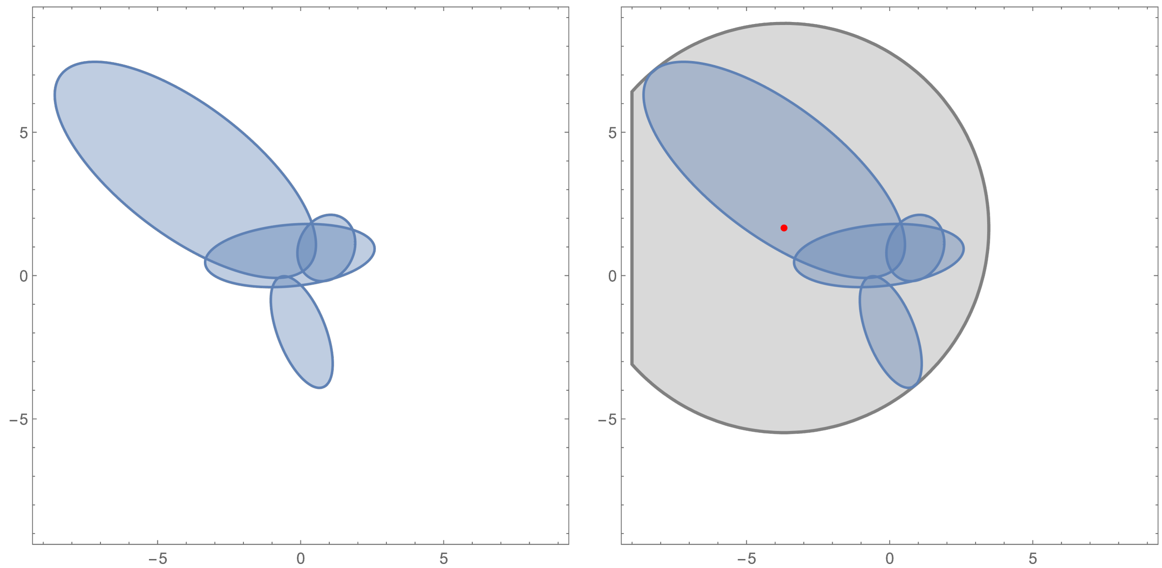
[1] This example is taken from [VB96, p. 58].

[2] We are implicitly assuming that both ellipses have nonempty interior.

Thus, we arrive at the SDP

$$\min \quad t \quad \text{s.t.} \quad \begin{pmatrix} \mathrm{Id} & -x_c \\ -x_c^T & \gamma \end{pmatrix} \preceq \tau_i \begin{pmatrix} A_i & b_i \\ b_i^T & c_i \end{pmatrix}, \quad i = 1, \ldots, p.$$
$$\tau_i \geq 0, \quad i = 1, \ldots, p.$$
$$\begin{pmatrix} \mathrm{Id} & x_c \\ x_c^T & t+\gamma \end{pmatrix} \succeq 0.$$

See file `lecture24.nb` for an implementation.



References

[BPT13] G. Blekherman, P. A. Parrilo, and R. R. Thomas. *Semidefinite optimization and convex algebraic geometry*, vol. 13 of MOS-SIAM Series on Optimization. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA; Mathematical Optimization Society, Philadelphia, PA, 2013.

[VB96] L. Vandenberghe, S. Boyd. *Semidefinite Programming*, SIAM Review, Vol. 38, No. 1, 49-95, 1996. `https://web.stanford.edu/~boyd/papers/pdf/semidef_prog.pdf`