MAT347/644, Spring 2026 **Renato Ghini Bettiol**

Lecture 3

1. Linear programs in 2 variables

A *linear program (LP)* in 2 variables $x = (x_1, x_2)$ is an optimization problem of the form

$$\begin{aligned} \min \quad c_1x_1 + c_2x_2 \quad \text{s.t.} \quad & a_{11}\, x_1 + a_{12}\, x_2 \le b_1, \\ & a_{21}\, x_1 + a_{22}\, x_2 \le b_2, \\ & \dots \\ & a_{m1}\, x_1 + a_{m2}\, x_2 \le b_m, \\ & x_1 \ge 0,\ x_2 \ge 0. \end{aligned}$$

Note that the above LP can be rewritten in matrix notation as

$$\begin{aligned} \min \quad c^T x \quad \text{s.t.} \quad & A\, x \le b, \\ & x \ge 0, \end{aligned} \tag{1}$$

where $x = (x_1, x_2)$, inequalities such as $v \le w$ between vectors are defined to mean the coordinate-wise inequalities between the corresponding entries $v_i \le w_i$, and $A = (a_{ij})$ is an $m \times 2$ matrix, $b = (b_i) \in \mathbb{R}^m$, where the indices have ranges $1 \le i \le m$ and $1 \le j \le 2$.

Recall that $(\cdot)^T$ denotes the *transpose* of a vector, or of a matrix. In particular, $x^T y$ is nothing but the dot product of the vectors x and y, also often written $x \cdot y$ or $\langle x, y\rangle$.

Exercise 1. Find the matrix A and vectors b, c so that the problem from Lecture 1:

$$\begin{aligned} \min \quad 1.00\, x_1 + 1.20\, x_2 \quad \text{s.t.} \quad & 8x_1 + 6x_2 \ge 11, \\ & 4x_1 + 12x_2 \ge 16, \\ & x_1 \ge 0,\ x_2 \ge 0, \end{aligned}$$

can be written in the above form (1). (Remember that $u \le v$ if and only if $-u \ge -v$.)

Exercise 2. How can you relate the *maximization* problem

$$\begin{aligned} \max \quad c^T x \quad \text{s.t.} \quad & A\, x \le b, \\ & x \ge 0, \end{aligned}$$

to (1)? We shall also refer to an optimization problem as above as a linear program (LP).

The set of points $x \in \mathbb{R}^2$ that satisfy the *constraints* $A\, x \le b$ and $x \ge 0$ is called the *feasible region*, and points in the feasible region are called *feasible solutions*. A feasible solution is called an *optimal solution* if it achieves the min/max of the *target function* $c^T x$. The feasible region of an LP in 2 variables is an intersection of finitely many half spaces $a_{i1}x_1 + a_{i2}x_2 \le b_i$ hence it is a convex polygon in $\mathbb{R}^2$. Note that it may be empty (problem is *infeasible*) or noncompact.

Exercise 3. Give examples of LPs with the following feasible regions:
(a) the empty set;
(b) the first quadrant;
(c) the horizontal strip $[0, +\infty) \times [0, 1]$;
(d) the square $[0, 1] \times [0, 1]$;
(e) the triangle with vertices $(0, 0)$, $(2, 0)$ and $(0, 3)$.

2. Brute force solution and geometric solution

In order to solve an LP in 2 variables, one may follow a very direct geometric method:

(i) Identify the feasible region $S \subset \mathbb{R}^2$, which is a polygon: more precisely, find the coordinates of all vertices (extremal points) of S;
(ii) Compute the target function at all vertices;
(iii) Order the results; the *smallest* value is the min.

As we shall see later, this is a rather crude and brue force approach, which would be extremely slow in larger problems. However, it is a first step in our journey to solving LPs.

Exercise 4. Implement the above strategy in the following LP:

$$\begin{aligned} \max \quad 4\,x_1 - 2\,x_2 \quad \text{s.t.} \quad & 2x_1 + 4x_2 \leq 12 \\ & x_1 + x_2 \leq 5 \\ & x_2 \leq 5/2 \\ & x_1 - x_2 \leq 4 \\ & x_1 \geq 0,\ x_2 \geq 0. \end{aligned}$$

What is the optimal solution? Repeat replacing max with min.

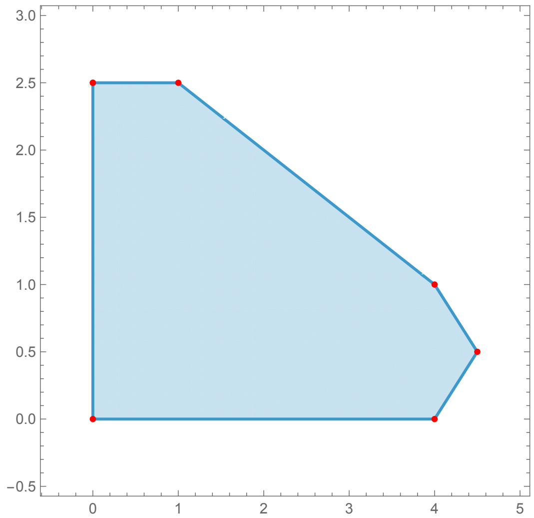
Exercise 5. Explain the procedure you used to find the vertices. How well would it scale if the number of sides of the polygon grows?

An improvement on the above is to find the optimal solution by considering levelsets

$$L_t = \{x \in \mathbb{R}^2 : c^T x = t\}$$

of the target function for varying $t \in \mathbb{R}$. The gradient of $x \mapsto c^T x$ is clearly the vector c^T, which is therefore orthogonal to the levelsets L_t. ‘Moving in’ levelset lines L_t from ‘infinity’ until they touch the feasible region, we can geometrically identify which vertex is the optimal solution, and then find its coordiantes by solving the equations that correspond to the lines intersecting at that vertex.

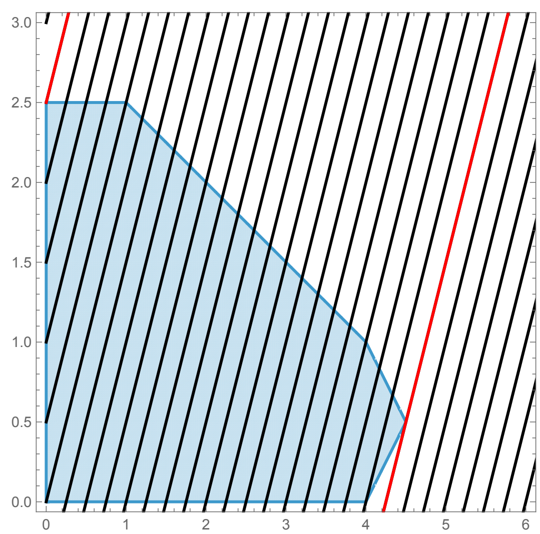
Solution to Exercise 4. The feasible region is as follows:



In order to find the vertices, we find all pairwise intersections of lines that define the boundary of the feasible region and discarding those solutions that are not feasible. Note that these lines correspond to the equations obtained from the constraints by replacing $\leq$ with $=$. Analyzing the given constraints, we conclude that the vertices are:

$$(4, 1),\ (1, 5/2),\ (9/2, 1/2),\ (0, 5/2),\ (4, 0),\ (0, 0).$$

In Exercise 5 we discuss how to systematize this process of finding vertices (for now, in dimension 2). Overlapping with levelsets, we find:



Thus, $\min c^T x = -5$ is achieved at $(0, 5/2)$, and $\max c^T x = 17$ is achieved at $(9/2, 1/2)$.

Solution to Exercise 5 (Method to find vertices)**.** Let us write the feasible region $S \subset \mathbb{R}^2$ as the set of $x \in \mathbb{R}^2$ such that $Ax \leq b$. Note that the matrix A and the vector b must encode *all* constraints, including $x \geq 0$.

The key insight is that $v \in S$ is a vertex if and only if $v \in S$ is a point where 2 linearly independent inequality constraints hold with equality. Thus, we proceed as follows: for each choice of 2 (linearly independent) inequalities

$$a_{i_1}^T x \leq b_{i_1} \quad \text{and} \quad a_{i_2}^T x \leq b_{i_2}$$

we transform them into equations

$$a_{i_1}^T x = b_{i_1} \quad \text{and} \quad a_{i_2}^T x = b_{i_2}$$

and find the unique solution $x_{(i_1,i_2)} \in \mathbb{R}^2$ to this system of linear equations using, e.g., row reduction (Gaussian elimination). Then, we must check if $x_{(i_1,i_2)} \in S$, which is done by plugging in $x_{(i_1,i_2)}$ into the remaining inequalities $a_j^T x \leq b_j$, for all $j \notin \{i_1, i_2\}$. If they are *all* satisfied, then $x_{(i_1,i_2)}$ is a vertex of S. If not, we discard $x_{(i_1,i_2)}$ and move on to the next choice of 2 linearly independent constraints.

Note that if S is determined by m linearly independent[1] inequalities, then the above process will require solving $\binom{m}{2}$ systems of linear equations, and checking if the solution satisfies the remaining $m - 2$ inequalities.

[1] Use row reduction on A to determine if there are 'redundant' constraints and remove them to work with linearly independent constraints.