Lecture 4

1. Convex geometry

A set $S \subset \mathbb{R}^n$ is *convex* if given any $x, y \in S$, the line segment $(1-t)x + ty$, $0 \leq t \leq 1$, joining x and y lies entirely in S.

Exercise 1. Use the definition of convexity given above to:

(i) give examples of sets that are convex and not convex;
(ii) classify the convex sets in $\mathbb{R}$;
(iii) prove that the intersection of two convex sets is convex;
(iv) decide if, in general, the union of convex sets is convex.

A set $S \subset \mathbb{R}^n$ is a convex *polyhedron* if it can be written in the form

$$S = \{x \in \mathbb{R}^n : Ax \leq b\}, \tag{1}$$

where $A = (a_{ij})$ is an $m \times n$ matrix, $b = (b_i) \in \mathbb{R}^m$, with indices $1 \leq i \leq m$ and $1 \leq j \leq n$. In other words, $S = \bigcap_{1 \leq i \leq m} H_i$ is the intersection of the half-spaces $H_i = \{x \in \mathbb{R}^n : a_i^T x \leq b_i\}$, where $a_i \in \mathbb{R}^n$ is the ith row of A.

Exercise 2. Show that half-spaces are convex and conclude that convex polyhedra are convex.

Exercise 3. Identify the convex polyhedra (1) where A and b are as follows:

(i) $A = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

(ii) $A = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $b = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

(iii) $A = \begin{pmatrix} 1 & -2 \\ 0 & 1 \\ 3 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$

(iv) $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

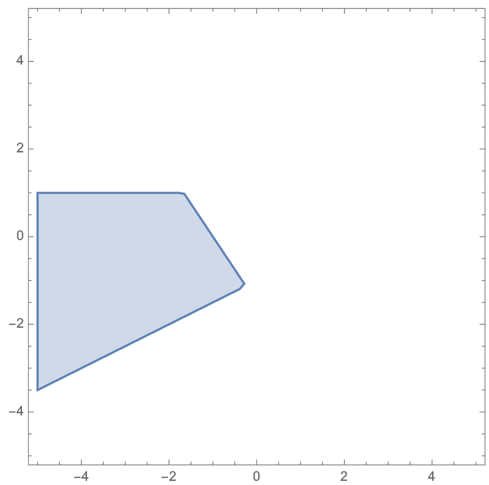
Solution to Exercise 2. Let $H = \{x \in \mathbb{R}^n : a^T x \le b\}$, where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$, be a half-space. If $x, y \in H$, then $a^T x \le b$ and $a^T y \le b$. Therefore, if $0 \le t \le 1$, we have

$$a^T((1-t)x + ty) = (1-t)\, a^T x + t\, a^T y \le (1-t)b + tb = b,$$

i.e., $(1-t)x+ty \in H$, which proves that H is convex. Since convex polyhedra are intersections of half-spaces, which are convex, it follows by Exercise 1 (iii) that convex polyhedra are convex.

Solution to Exercise 3. The polyhedra are as follows:

(i) Interval $[-2, 3]$;
(ii) Empty set $\emptyset$;
(iii) $x_1 - 2x_2 \le 2$, $x_2 \le 1$, $3x_1 + 2x_2 \le -3$



(iv) Unit cube $[0, 1] \times [0, 1] \times [0, 1]$.

A point $v \in S$ in a convex set S is called *extremal* if $v = (1-t)x + ty$ with $x, y \in S$ and $0 \leq t \leq 1$ implies that either $t = 0$ or $t = 1$. In other words, v is extremal if it *cannot* be placed in the *interior* of any line segment with endpoints in S.

Exercise 4. Determine the extremal points of the following convex sets:
(i) A bounded polyhedron $S \subset \mathbb{R}^n$
(ii) The unit ball $B = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$

A *convex combination* of the points $x_1, \ldots, x_r \in \mathbb{R}^n$ is any point of the form

$$c_1 x_1 + \cdots + c_r x_r \in \mathbb{R}^n,$$

where $c_1, \ldots, c_r \in \mathbb{R}$ satisfy $\sum_{i=1}^{r} c_i = 1$ and $c_i \geq 0$ for all $1 \leq i \leq r$. The set of all convex combinations of $x_1, \ldots, x_r$ is called the *convex hull* of $x_1, \ldots, x_r$, and denoted $\operatorname{conv}(x_1, \ldots, x_r)$.

Exercise 5. Prove that $\operatorname{conv}(x_1, \ldots, x_r)$ is convex.

Exercise 6. What is the convex hull of 2 points in $\mathbb{R}^n$?

Exercise 7. What is the convex hull of n points in $\mathbb{R}^2$?