Lecture 5

1. Extremal points

Recall that $S \subset \mathbb{R}^n$ is *convex* if given any $x, y \in S$, the line segment $(1-t)x + ty$, $0 \leq t \leq 1$, joining x and y lies entirely in S. A set $S \subset \mathbb{R}^n$ is *bounded* if there exists $R > 0$ such that all points in S are at distance at most R from $0 \in \mathbb{R}^n$, that is, for all $x \in S$, $\|x\| \leq R$.

A point $v \in S$ in a convex set is called *extremal* if $v = (1-t)x + ty$ with $x, y \in S$ and $0 \leq t \leq 1$ implies that either $t = 0$ or $t = 1$. In other words, v is extremal if it *cannot* be placed in the *interior* of any line segment with endpoints in S.

Exercise 1. Determine the extremal points of the following convex sets:

(i) A bounded polyhedron $S \subset \mathbb{R}^n$

(ii) The unit ball $B = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$

A *convex combination* of the points $x_1, \dots, x_r \in \mathbb{R}^n$ is a point of the form

$$c_1 x_1 + \cdots + c_r x_r \in \mathbb{R}^n,$$

where $c_1, \dots, c_r \in \mathbb{R}$ satisfy $\sum_{i=1}^{r} c_i = 1$ and $c_i \geq 0$ for all $1 \leq i \leq r$. The set of all convex combinations of $x_1, \dots, x_r$ is called the *convex hull* of $x_1, \dots, x_r$, and denoted $\operatorname{conv}(x_1, \dots, x_r)$.

Exercise 2. Prove that $\operatorname{conv}(x_1, \dots, x_r)$ is convex.

Exercise 3. What is the convex hull of 2 points in $\mathbb{R}^n$?

Exercise 4. What is the convex hull of n points in $\mathbb{R}^2$?

The following are foundational statements that we will use but not prove. (You might want to think about how you would prove them.)

Theorem 1. *A polyhedron is bounded if and only if it does not contain a line.*

Theorem 2 (Krein-Milman, baby version)**.** *A bounded polyhedron coincides with the convex hull of its vertices (i.e., its extremal points).*

By the above, 'determining' a bounded polyhedron is the same as 'determining' its vertices. In order to do this using as input the description $S = \{x \in \mathbb{R}^n : Ax \leq b\}$ of a polyhedron as an intersection of half-spaces $a_i^T x \leq b_i$, recall that a feasible solution $x \in S$ is a vertex if it lies in the intersection of n of the above half-spaces, provided they are linearly independent (hence their intersection is a single point). This leads us to the following:

Theorem 3. *Consider the polyhedron $S = \{x \in \mathbb{R}^n : Ax \leq b\}$, where A is an $m \times n$ matrix and $b \in \mathbb{R}^m$. A point $v \in S$ is a vertex of S if and only if there exist n linearly independent inequality constraints of S that hold with equality at v, i.e., there exist $i_1, \dots, i_n \in \{1, \dots, m\}$ such that $a_{i_1}^T v = b_{i_1}$, $\dots$, $a_{i_n}^T v = b_{i_n}$ and $\{a_{i_1}, \dots, a_{i_n}\}$ are linearly independent.*

The above yields a method to find all vertices of a polyhedron

$$S = \{x \in \mathbb{R}^n : Ax \le b\},$$

namely one proceeds as follows. `For` each[1] subset $\{i_1, \dots, i_n\}$ of $\{1, \dots, m\}$, `do`:

(i) Check if $a_{i_1}, \dots, a_{i_n}$ are linearly independent (if `NO`, then `STOP`);
(ii) Compute the unique solution $v \in \mathbb{R}^n$ to $a_{i_1}^T v = b_{i_1}, \dots, a_{i_n}^T v = b_{i_n}$;
(iii) If $v \in S$, i.e., $Av \le b$, then v is a vertex. If not, then it is not a vertex.

Running the above `for` loop through all subsets of $\{1, \dots, m\}$ and collecting the resulting vertices, one obtains the complete list of vertices of S. In particular, this proves that a polyhedron only has finitely many vertices.

This algorithm is implemented in the Mathematica notebook `vertices.nb`

Exercise 5. Find all vertices of the polyhedron $S = \{x \in \mathbb{R}^n : Ax \le b\}$ where

(i) $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(ii) $A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 0 \\ 3 \end{pmatrix}$

2. Linear programs in any number of variables

A general *linear program (LP)* in n variables $x = (x_1, x_2, \dots, x_n)$ is an optimization problem of the form

$$\begin{aligned} \min \quad c_1x_1 + c_2x_2 + \cdots + c_nx_n \quad \text{s.t.} \quad & a_{11}\,x_1 + a_{12}\,x_2 + a_{1n}\,x_n \le b_1, \\ & a_{21}\,x_1 + a_{22}\,x_2 + a_{2n}\,x_n \le b_2, \\ & \dots \\ & a_{m1}\,x_1 + a_{m2}\,x_2 + a_{mn}\,x_n \le b_m. \end{aligned} \tag{1}$$

The above constraints might have been obtained from linear constraints with $\le$, $=$, or $\ge$, using the elementary tricks we discussed in lecture, and might (or might not) include the nonnegativity constraints $x_1 \ge 0, \dots x_n \ge 0$. Recall that maximization problems reduce to the above as well.

Note that the above LP can be rewritten in matrix notation as

$$\min \quad c^T x \quad \text{s.t.} \quad A\,x \le b, \tag{2}$$

[1] Note there are $\binom{m}{n}$ such subsets.

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$, $b = (b_1, \ldots, b_m) \in \mathbb{R}^m$, and

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

The rows of A are denoted $a_i = (a_{i1}, a_{i2}, \ldots, a_{in})$, for $1 \leq i \leq m$. These conventions are as in earlier lectures; in particular, the indices have ranges $1 \leq i \leq m$ and $1 \leq j \leq n$.

Finally, the following statement explains the relevance of extremal points:

Theorem 4. *If the optimization problem* (2) *is* feasible *and* bounded*, i.e., the polyhedron $S = \{x \in \mathbb{R}^n : Ax \leq b\}$ is nonempty and bounded, then there exists an extremal point $v \in S$ which is an optimal solution.*