MAT347/644, Spring 2026 **Renato Ghini Bettiol**

Lecture 6

1. Linear programs in any number of variables

A general *linear program (LP)* in n variables $x = (x_1, x_2, \ldots, x_n)$ is an optimization problem of the form

$$\begin{aligned} \min \quad c_1x_1 + \cdots + c_nx_n \quad \text{s.t.} \quad & a_{11}\,x_1 + a_{12}\,x_2 + \cdots + a_{1n}\,x_n \le b_1, \\ & a_{21}\,x_1 + a_{22}\,x_2 + \cdots + a_{2n}\,x_n \le b_2, \\ & \ldots \\ & a_{m1}\,x_1 + a_{m2}\,x_2 + \cdots + a_{mn}\,x_n \le b_m. \end{aligned} \tag{1}$$

The above constraints might have been obtained from linear constraints with $\le$, $=$, or $\ge$, using the elementary tricks we discussed in lecture, and might (or might not) include the nonnegativity constraints $x_1 \ge 0, \ldots x_n \ge 0$. Recall that maximization problems reduce to the above as well.

Note that the above LP can be rewritten in matrix notation as

$$\min \quad c^T\,x \quad \text{s.t.} \quad A\,x \le b, \tag{2}$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$, $b = (b_1, \ldots, b_m) \in \mathbb{R}^m$, and

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

The rows of A are denoted $a_i = (a_{i1}, a_{i2}, \ldots, a_{in})$, for $1 \le i \le m$. These conventions are as in earlier lectures; in particular, the indices have ranges $1 \le i \le m$ and $1 \le j \le n$.

Finally, the following statement explains the relevance of extremal points:

Theorem 1. *If the optimization problem* (2) *is* feasible *and* bounded*, i.e., the polyhedron $S = \{x \in \mathbb{R}^n : Ax \le b\}$ is nonempty and bounded, then there exists an extremal point $v \in S$ which is an optimal solution.*

2. Slack variables and equational formulation of LP

The LP of the form (2) has feasible set

$$S = \{x \in \mathbb{R}^n : Ax \le b\}. \tag{3}$$

In order to solve a general LP of the form (2), we will first rewrite it in the *equational form*

$$\begin{aligned} \min \quad c^T\,x \quad \text{s.t.} \quad & A\,x = b, \\ & x \ge 0. \end{aligned} \tag{4}$$

This can be achieved by introducing *slack variables*, which will 'enlarge' the vector x with new entries. The key observation is:

$a_i^T \cdot x \le b_i$ if and only if there exists $s_i \ge 0$ such that $a_i^T x + s_i = b_i$.

Augmenting A and b using this observation turns any LP in standard form (2) into an LP in equational form (4).

Let us analyze a very simple example in dimension 1. As we have seen before, the only convex subsets in $\mathbb{R}$ are intervals. Up to a translation, every bounded convex subset is therefore of the form $S = [0, L]$ for some $L > 0$. An LP in one variable with this feasible region looks like

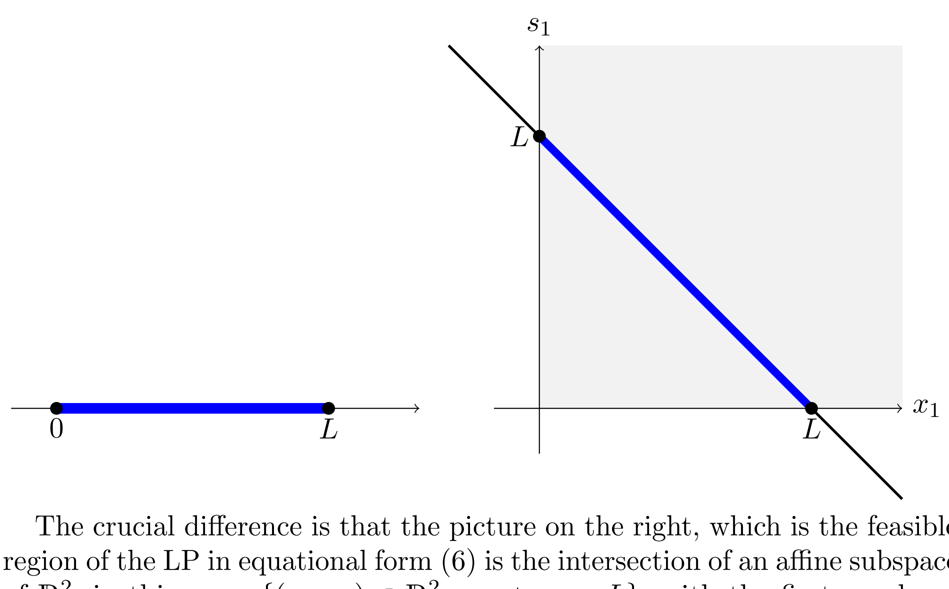
$$\begin{aligned} \min \quad c\,x_1 \quad \text{s.t.} \quad & x_1 \leq L \\ & x_1 \geq 0. \end{aligned} \tag{5}$$

To write it in the form (2), we replace '$x_1 \geq 0$' with '$-x_1 \leq 0$' and use $A = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $x = (x_1)$, and $b = \begin{pmatrix} L \\ 0 \end{pmatrix}$, so that S in (3) becomes $S = [0, L]$. To rewrite (5) in equational form, we must introduce one slack variable s_1, so that $x_1 + s_1 = L$ and $s_1 \geq 0$. Thus, our LP becomes

$$\begin{aligned} \min \quad c\,x_1 \quad \text{s.t.} \quad & \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ s_1 \end{pmatrix} = L, \\ & x_1 \geq 0,\ s_1 \geq 0, \end{aligned} \tag{6}$$

which is of the form (4) with $A = \begin{pmatrix} 1 & 1 \end{pmatrix}$, $b = L$, and $x = \begin{pmatrix} x_1 \\ s_1 \end{pmatrix}$. Note that (5) and (6) are equivalent problems; in particular, their optimal values (which exist by Theorem 1) are the same. In fact, if $c \geq 0$, then $x_1 = 0$ is an optimal solution and if $c \leq 0$, then $x = L$ is an optimal solution. If the inequality for c is strict, then that is the unique solution.

Geometrically, the feasible regions in (5) and (6) look very similar:



The crucial difference is that the picture on the right, which is the feasible region of the LP in equational form (6) is the intersection of an affine subspace of $\mathbb{R}^2$, in this case, $\{(x_1, s_1) \in \mathbb{R}^2 : x_1 + s_1 = L\}$, with the first quadrant $\{(x_1, s_1) \in \mathbb{R}^2 : x_1 \geq 0, s_1 \geq 0\}$. This 'standardized' way of working with the feasible region will be key to locate its extremal points. (This might not seem like a big deal here but will be a huge advantage in higher dimensions.)

Exercise 1. Add slack variables to the following LPs in standard form, transforming them in equational form (4). What are x, A, b, c in each case?

a) $$\begin{aligned} \min \quad x_1 - 2x_2 \quad \text{s.t.} \quad & x_1 + x_2 \leq 1, \\ & x \geq 0, \end{aligned}$$

b) $$\begin{aligned} \min \quad 2x_1 - 3x_2 + x_3 \quad \text{s.t.} \quad & x_1 + 3x_2 + x_3 \leq 5, \\ & 4x_1 - x_2 - x_3 \leq 2 \\ & 9x_1 + 2x_2 + 7x_3 \leq 3 \\ & x \geq 0, \end{aligned}$$

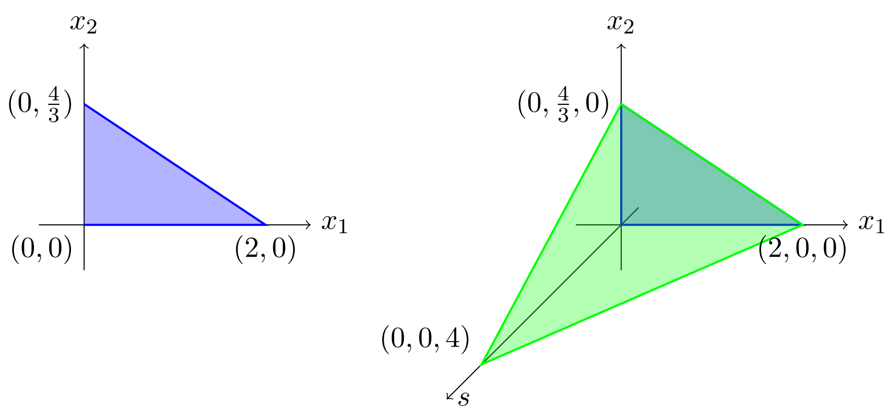
For an example with two variables that requires adding one slack variable, consider

$$\begin{aligned} \min \quad x_1 + x_2 \quad \text{s.t.} \quad & 2x_1 + 3x_2 \leq 4, \\ & x \geq 0, \end{aligned}$$

which, in equational form, becomes

$$\begin{aligned} \min \quad x_1 + x_2 \quad \text{s.t.} \quad & 2x_1 + 3x_2 + s_1 = 4, \\ & x \geq 0, \end{aligned}$$

where $x = (x_1, x_2)$ in the first problem, while $x = (x_1, x_2, s_1)$ in the second problem. Note that the feasible regions for the above LPs are triangles, but the first is contained in the (x_1, x_2)-plane while the second lives in the (x_1, x_2, s_1)-space and projects onto the first one via $(x_1, x_2, s_1) \mapsto (x_1, x_2)$.



Exercise 2. If the feasible region was a quadrilateral instead of a triangle, how many additional dimensions would be needed to write it in equational form? What about for polygons with k sides?

After adding slack variables, if some variables x_i in the original LP in standard form were not constrained to be nonnegative, we can replace them by $x_i = y_i - z_i$ where $y_i \geq 0$ and $z_i \geq 0$. This second step further augments x, A, b, c from the original ones.

Exercise 3. Make the appropriate changes to the following LP to transform it into equational form (4).

$$\min \quad 2x_1 + x_2 \quad \text{s.t.} \quad \begin{aligned} x_1 + x_2 &\le 1, \\ x_1 &\le 1, \end{aligned}$$

The solution to the above exercise is a new LP with 6 variables. Plot the feasible region of the original LP and think about the minimum number of 'extra' dimensions that are added in this process. Do we need to go from 2 to 6 dimensions necessarily?

After performing the above steps (adding slack variables to replace $\le$ with $=$) and, if needed, exchanging variables so that all of them are constrained to be nonnegative, we arrive at an LP in equational form (4). In other words, the feasible set (3) is now written as the intersection of an affine space with the nonnegative orthant in $\mathbb{R}^n$, i.e.,

$$S = \{x \in \mathbb{R}^n : Ax = b\} \cap \{x \in \mathbb{R}^n : x \ge 0\}. \tag{7}$$

We shall further assume that the rows a_i of A are linearly independent, i.e., $\operatorname{rank} A = m$, as we may discard redundant (linearly dependent) constraints.

Exercise 4. Suppose that the feasible region of an LP is $Ax = b$ where

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 1 \\ 3 & 3 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 7 \\ 8 \\ 15 \end{pmatrix}$$

appears in the formulation of an LP in equational form. Remove the appropriate row(s) so that the remaining rows are linearly independent. What are the new A and b?

To simplify notation, we will continue using the convention that the matrix A in an LP has dimensions $m \times n$, after possibly being augmented and having certain rows removed, according to the procedures described above.

3. Basic feasible solutions

Adding slack variables $s = (s_1, s_2, s_3)$ to the LP in Exercise 1 b), we arrive at the equational formulation of the feasible set

$$\begin{aligned} &x_1 + 3x_2 + x_3 + s_1 = 5, \\ &4x_1 - x_2 - x_3 + s_2 = 2 \\ &9x_1 + 2x_2 + 7x_3 + s_3 = 3 \\ &x \ge 0, \ s \ge 0 \end{aligned}$$

Note that this system of equations has an *obvious* solution $x = 0$, $s = (5, 2, 3)$, obtained by 'zeroing' out the original variables x and 'leaving' everything (from the right-hand side) in the slack variables s. This is an example of a *basic feasible solution*, corresponding to choosing s_1, s_2, s_3 as *basic variables*, i.e., choosing the last three columns of the augmented matrix

$$\begin{pmatrix} 1 & 3 & 1 & 1 & 0 & 0 \\ 4 & -1 & -1 & 0 & 1 & 0 \\ 9 & 2 & 7 & 0 & 0 & 1 \end{pmatrix}$$

Of course, the same can be done every time we have an LP with one slack variable for each constraint, i.e., every time the augmented matrix has an $m \times m$ identity block appended on the right, as the one above.

More on this next time!