Lecture 7

1. Basic feasible solutions

Recall that, by (i) introducing slack variables and (ii) replacing unconstrained variables by the difference of two nonnegative variables, we may rewrite an LP in the *equational form*

$$\min \quad c^T x \quad \text{s.t.} \quad \begin{aligned} Ax &= b, \\ x &\geq 0. \end{aligned} \tag{1}$$

Moreover, by removing any rows of the $m \times n$ matrix A that give redundant constraints, we shall also assume that $\operatorname{rank}(A) = m$. (Typically, $m < n$.)

A point $v \in \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$ is a *basic feasible solution* of (1) if there is a subset $B \subset \{1, \dots, n\}$ of size $|B| = m$ such that the matrix A_B made of the columns of A indexed by B is invertible and $v_j = 0$ for all $j \notin B$.

Exercise 1. Writing the LP

$$\min \quad -x_1 + x_2 \quad \text{s.t.} \quad \begin{aligned} x_1 + 2x_2 &\leq 4 \\ 3x_1 + x_2 &\leq 6 \\ x &\geq 0, \end{aligned}$$

in equational form (1), we find $A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$.

(i) Is $v = (2, 0, 2, 0)$ a basic feasible solution? What about $v = (1, 0, 3, 3)$?
(ii) Find a basic feasible solution with $B = \{3, 4\}$ and $B = \{1, 2\}$.

The importance of basic feasible solutions is explained by the following:

Theorem 1. *A point $v \in \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$ is a basic feasible solution of* (1) *if and only if v is an extremal point (i.e., a vertex) of the polyhedron $\{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$.*

Theorem 2. *If the LP given in* (1) *is bounded and feasible, then there exists a basic feasible solution which is an optimal solution.*

A subset $B \subset \{1, \dots, n\}$ with $|B| = m$ such that A_B is invertible is called a *basis*. If, in addition, the unique $x_B \in \mathbb{R}^n$ such that $Ax_B = b$ and $x_j = 0$ for all $j \notin B$ satisfies $x_B \geq 0$, then B is called a *feasible basis*.

Exercise 2. Show that $B = \{1, 3\}$ is a feasible basis for the LP in Exercise 1. Is $B = \{1, 4\}$ a feasible basis?

2. Simplex method

The *simplex method* is an algorithm to find a basic feasible solution that is an optimal solution to an LP, or conclude that the LP is unbounded or infeasible. Essentially it is an iterative procedure in which we switch between basic feasible solutions x_B by switching the feasible basis B in a way that improves the value of the target function, until it becomes optimal (or unbounded). Let us examine the following example, obtained by adding one additional constraint to the LP in Exercise 1:

$$\begin{aligned} \min \quad -x_1 + x_2 \quad \text{s.t.} \quad & x_1 + 2x_2 \le 4 \\ & 3x_1 + x_2 \le 6 \\ & x_1 \le 2 \\ & x \ge 0. \end{aligned}$$

Introducing slack variables x_3, x_4, x_5, we rewrite it as

$$\begin{aligned} \min \quad -x_1 + x_2 \quad \text{s.t.} \quad & x_1 + 2x_2 + x_3 = 4 \\ & 3x_1 + x_2 + x_4 = 6 \\ & x_1 + x_5 = 2 \\ & x \ge 0. \end{aligned}$$

Observe that the corresponding 3×5 matrix A and b are given by

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix},$$

and $\operatorname{rank} A = 3$.

Exercise 3. Is $B = \{3, 4, 5\}$ a feasible basis for the above LP? What is the corresponding basic feasible solution?

Solution to Exercise 3. Clearly, the only $x = (0, 0, x_3, x_4, x_5)$ that satisfies $Ax = b$ is $x_B = (0, 0, 4, 6, 2)$. Since this $x = x_B$ satisfies $x \ge 0$, we conclude B is a feasible basis and x_B is the corresponding basic feasible solution.

Note that, in the above case of $B = \{3, 4, 5\}$, A_B is the identity matrix. Let us now switch to a new $B = \{1, 3, 4\}$. A simple computation gives

$$A_B^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -3 \end{pmatrix}, \quad A_B^{-1}A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -3 \end{pmatrix}, \quad A_B^{-1}b = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix},$$

which allows us to write $\{x \in \mathbb{R}^n : Ax = b\} = \{x \in \mathbb{R}^n : A_B^{-1}Ax = A_B^{-1}b\}$ using parameters x_j with $j \notin B$, i.e., using parameters (x_2, x_5). Namely, the system $A_B^{-1}Ax = A_B^{-1}b$ reads

$$\begin{aligned} x_1 + \qquad\qquad\quad x_5 &= 2 \\ 2x_2 + x_3 \qquad - x_5 &= 2 \\ x_2 \qquad + x_4 - 3x_5 &= 0, \end{aligned}$$

and hence

$$\begin{aligned} x_1 &= 2 \qquad\quad - x_5 \\ x_3 &= 2 - 2x_2 + x_5 \\ x_4 &= \quad\;\; - x_2 + 3x_5 \end{aligned}$$

Plugging these in to the target function we find that it becomes

$$-x_1 + x_2 = -(2 - x_5) + x_2 = -2 + x_2 + x_5$$

The above is clearly ≥ -2 because $x \geq 0$. Thus, a feasible solution with $x_2 = x_5 = 0$ attains the minimum of this LP. From the above equations, one easily finds that $x = (2, 0, 2, 0, 0)$ is such a basic feasible solution, hence an optimal solution to the given LP.