

Lecture 8

1. SIMPLEX METHOD

The *simplex method* is an algorithm to find an optimal solution to LPs, consisting of steps in which we move from one feasible basis  $B \subset \{1, \dots, n\}$  to another  $B' = B \cup \{i\} \setminus \{j\}$  by adding an *entering basic variable*  $x_i$  and removing a *departing basic variable*  $x_j$ , in such a way that we improve the value of the target function  $c^T x$ . Recall that  $B \subset \{1, \dots, n\}$ ,  $|B| = m \leq n$ , is a *feasible basis* for an LP with feasible set in equational form

$$S = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\},$$

where  $A$  is an  $m \times n$  matrix of rank  $m$ , if  $A_B$  is invertible and the unique  $x_B \in \mathbb{R}^n$  such that  $Ax_B = b$  and  $x_j = 0$  for all  $j \notin B$  satisfies  $x_B \geq 0$ . Each feasible basis  $B$  corresponds to a vertex of  $S$ , so the above corresponds to moving from one vertex to another.

As an example, let us consider

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix}, \quad c = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

so that the LP is given by

$$\begin{aligned} \min \quad & -x_1 - x_2 \quad \text{s.t.} \quad x_1 + 2x_2 + x_3 = 4 \\ & 3x_1 + x_2 + x_4 = 6 \\ & x_1 + x_5 = 2 \\ & x \geq 0 \end{aligned}$$

An “obvious” feasible basis is  $B = \{3, 4, 5\}$ , with  $x_B = (0, 0, 4, 6, 2)$ . We represent this with the following *tableau*:

		$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	
(1)	$x_3$	1	2	1	0	0	4
	$x_4$	3	1	0	1	0	6
	$x_5$	1	0	0	0	1	2
	$z$	1	1	0	0	0	0

The bottom row indicates that the target function  $z = -x_1 - x_2$  takes value 0 at the basic feasible solution  $(0, 0, 4, 6, 2)$ , which is written as  $z + x_1 + x_2 = 0$ .

In order to select an *entering variable*, we pick the smallest index  $i \notin B$  such that the coefficient in the  $x_i$  column of the tableau is  $\geq 0$ ; in this case, the entering variable is  $x_1$ .

**Exercise 1.** Show that by increasing  $x_1$  from 0 to a positive value, we will decrease the target function.

To select the *departing variable*, we see how large can we make  $x_1$  while staying in the feasible set. Namely, let us look at the constraints  $x_i \geq 0$ ,  $i \in B$ , in terms of  $x_1$  and see which would be violated first if we increase  $x_1$

from 0 to a positive quantity (keeping in mind that  $x_2 = 0$  since  $x_2$  is not entering as a basic variable):

$$\begin{aligned} 0 \leq x_3 &= 4 - x_1 - 2x_2 \\ 0 \leq x_4 &= 6 - 3x_1 - x_2 \\ 0 \leq x_5 &= 2 - x_1 \end{aligned}$$

Since  $x_i \geq 0$ , we see from the above inequalities that  $x_1 \leq 4$ ,  $x_1 \leq 2$  and  $x_1 \leq 2$ , respectively. These quantities are sometimes called  $\theta$ -ratios, and are obtained dividing the coefficients on the last column of the tableau by the coefficients in the column of the entering variable:

$$\theta(x_3) = 4, \quad \theta(x_4) = 2, \quad \theta(x_5) = 2.$$

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	
$x_3$	1	2	1	0	0	4
$x_4$	3	1	0	1	0	6
$x_5$	1	0	0	0	1	2
$z$	1	1	0	0	0	0

The choice of departing variable is now arbitrary between  $x_4$  and  $x_5$ , since their vanishing happens before  $x_3$  vanishes, i.e., they have the smallest  $\theta$ -ratio  $\theta(x_4) = \theta(x_5) = 2$ .

Let us proceed selecting  $x_4$ , so the new feasible basis is  $B' = \{1, 3, 5\}$ . The corresponding tableau is obtained by row operations which correspond to multiplying by the matrix  $A_{B'}$ , so that the columns of  $x_i$  with  $i \in B'$  are the columns of an  $m \times m$  identity matrix:

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	
$x_1$	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	2
$x_3$	0	$\frac{5}{3}$	1	$-\frac{1}{3}$	0	2
$x_5$	0	$-\frac{1}{3}$	0	$-\frac{1}{3}$	1	0
$z$	0	$\frac{2}{3}$	0	$-\frac{1}{3}$	0	-2

Note that the value of the target function decreased from  $z = 0$  to  $z = -2$  at this new basic feasible solution  $x = (2, 0, 2, 0, 0)$ . However, we have not yet arrived at the minimum, since the coefficient of  $x_2$  is still positive. So we take  $x_2$  to be the new entering variable!

To select a departing variable, let us again check that the constraints  $x_i \geq 0$  remain satisfied. From the last tableau, we have:

$$\begin{aligned} 0 \leq x_1 &= 2 - \frac{1}{3}x_2 - \frac{1}{3}x_4 \\ 0 \leq x_3 &= 2 - \frac{5}{3}x_2 + \frac{1}{3}x_4 \\ 0 \leq x_5 &= \frac{1}{3}x_2 + \frac{1}{3}x_4 \end{aligned}$$

Keeping in mind that  $x_4 = 0$  since  $x_4$  is not entering as a basic variable, the first inequality gives  $x_2 \leq 6$  and the second gives  $x_2 \leq \frac{6}{5}$ , while the last does not yield any constraint on  $x_2$ . In other words, we only consider the  $\theta$ -ratios

that are  $\geq 0$  and discard those of the form  $0/a$  if  $a < 0$ , such as  $0/(-\frac{1}{3})$ , since they do not yield any constraints:

$$\theta(x_1) = 6, \quad \theta(x_3) = \frac{6}{5}.$$

Thus, we set  $x_3$  as the departing variable, enforcing the strongest constraint on how large  $x_2$  can be when we increase it from 0 to a positive quantity (i.e., it has the smallest  $\theta$ -ratio).

Performing the row operations to move towards the new feasible basis  $B'' = \{1, 2, 5\}$ , i.e., multiplying by  $A_{B''}$ , we arrive at the tableau

$$(3) \quad \begin{array}{c|ccccc|c} & x_1 & x_2 & x_3 & x_4 & x_5 & \\ \hline x_1 & 1 & 0 & -\frac{1}{5} & \frac{2}{5} & 0 & \frac{8}{5} \\ x_2 & 0 & 1 & \frac{3}{5} & -\frac{1}{5} & 0 & \frac{6}{5} \\ x_5 & 0 & 0 & \frac{1}{5} & -\frac{2}{5} & 1 & \frac{2}{5} \\ \hline z & 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & 0 & -\frac{14}{5} \end{array}$$

Since there are no positive entries in the target row, the above tableau corresponds to an *optimal* solution  $x = (\frac{8}{5}, \frac{6}{5}, 0, 0, \frac{2}{5})$  of the LP, where the target function achieves its minimum  $z = -\frac{14}{5}$ . In other words, by the above tableau, the target function is given by:

$$z = -\frac{14}{5} + \frac{2}{5}x_3 + \frac{1}{5}x_4,$$

so increasing either  $x_3$  or  $x_4$  from the current value 0 would increase the value of the target function, of which we are seeking the minimum. We have thus found that minimum!

**Exercise 2.** Solve the LP with same constraints and target function, but with ‘min’ replaced with ‘max’.

**Exercise 3.** Write code in your favorite programming language to produce tableaus as the above, for any choice of feasible basis. (Remember to handle possible exceptions, such as a choice of indices not being a feasible basis!)