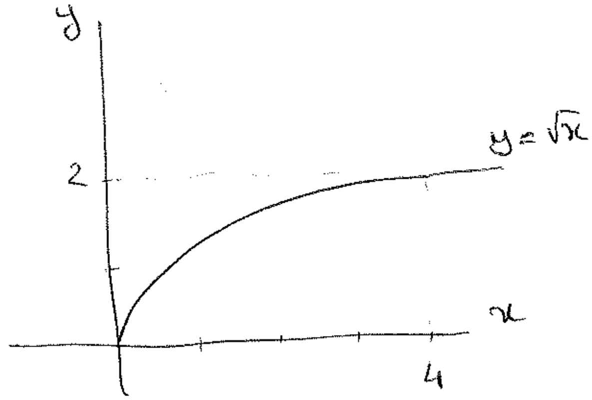
Homework 1 solutions

1. The base of a solid is the region between the x-axis, $y=\sqrt{x}$, and $x=4$. Each cross section perpendicular to the x-axis is a semicircle w/ diameter running along the base. What is the volume?



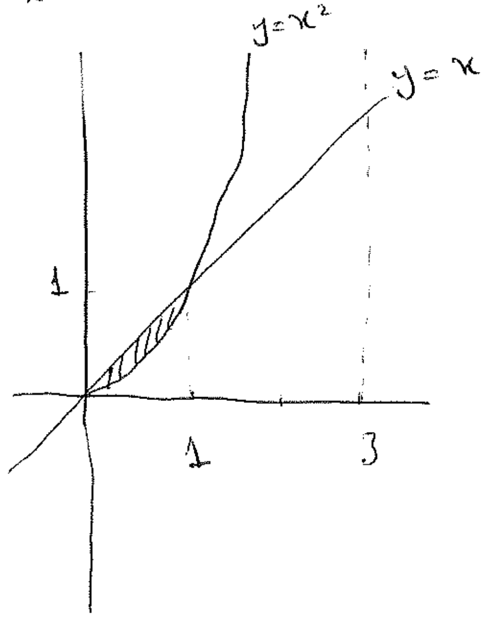
Radius of ~~each~~ semicircle at x is $\sqrt{x}/2$, so

$$A(x) = \frac{1}{2}\pi\left(\frac{\sqrt{x}}{2}\right)^2 = \frac{\pi x}{8}$$

$$V = \int_0^4 A(x)\,dx = \int_0^4 \frac{\pi x}{8}\,dx$$

$$= \frac{\pi x^2}{16}\Big|_0^4 = \pi$$

2. Find the volume of the solid obtained by revolving the region bounded by the line $y=x$ and the parabola $y=x^2$ about the line $x=3$.



Solving for $x^2=x$, we get either $x=0$ or $x=1$, so the line and the curve intersect at $(0,0)$ and $(1,1)$.

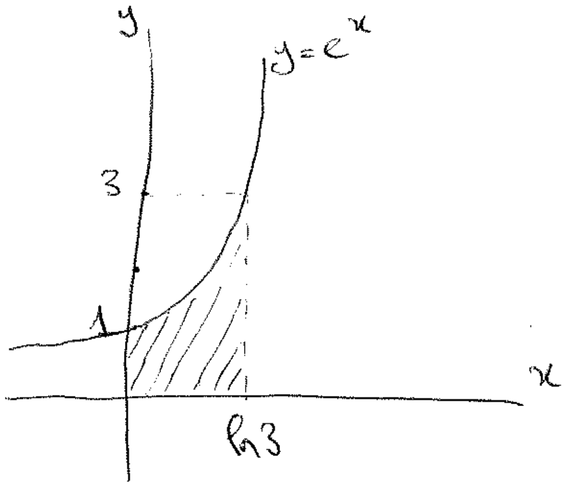
$$V = \int_0^1 \pi\left(R(y)^2 - r(y)^2\right)dy$$

$$= \int_0^1 \pi\left((3-y)^2-(3-\sqrt{y})^2\right)dy$$

$$= \int_0^1 \pi\left(9-6y+y^2-9+6\sqrt{y}-y\right)dy$$

$$= \int_0^1 \pi\left(y^2-7y+6\sqrt{y}\right)dy = \pi\left(\frac{y^3}{3}-\frac{7y^2}{2}+6\frac{y^{3/2}}{3/2}\right)\Big|_0^1 = \frac{5}{6}\pi$$

3. Find the volume of the solid obtained by revolving the region bounded by $y = e^x$, $x = 0$, $y = 0$, and $x = \ln 3$ about the x-axis.



$$V = \int_0^{\ln 3} \pi R(x)^2 \, dx$$

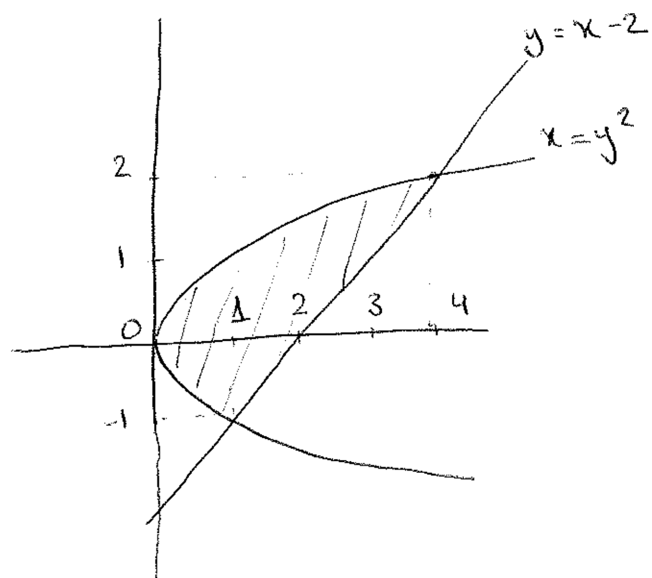
$$= \int_0^{\ln 3} \pi (e^x)^2 \, dx$$

$$= \int_0^{\ln 3} \pi e^{2x} \, dx$$

$$= \frac{\pi e^{2x}}{2} \Big|_0^{\ln 3}$$

$$= \frac{\pi}{2}(9 - 1) = 4\pi$$

4. Find the volume of the solid obtained by revolving the region bounded by $x = y^2$ and $y = x - 2$ about the y-axis



Solving for intersections:

$y^2 = y + 2 \Rightarrow y = 2$ or $y = -1$

$\Rightarrow$ the intersections are $(4, 2)$ and $(1, -1)$.

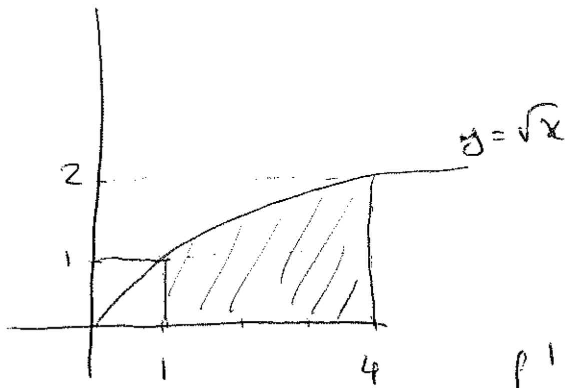
$$V = \int_{-1}^{2} \pi \left(R(y)^2 - r(y)^2 \right) dy$$

$$= \int_{-1}^{2} \pi \left((y+2)^2 - (y^2)^2 \right) dy$$

$$= \int_{-1}^{2} \pi \left(y^2 + 4y + 4 - y^4 \right) dy$$

$$= \pi \left(\frac{y^3}{3} + 2y^2 + 4y - \frac{y^5}{5} \right) \Big|_{-1}^{2} = \frac{72}{5}\pi$$

5. Find the volume of the solid obtained by revolving the region bounded by $y=\sqrt{x}$, $x=1$, $y=0$, and $x=4$ about the y-axis.



$$V = \int_0^2 \pi\left(R(y)^2 - r(y)^2\right)dy$$

$$= \int_0^1 \pi(4^2-1^2)\,dy + \int_1^2 \pi\left(4^2-(y^2)^2\right)dy$$

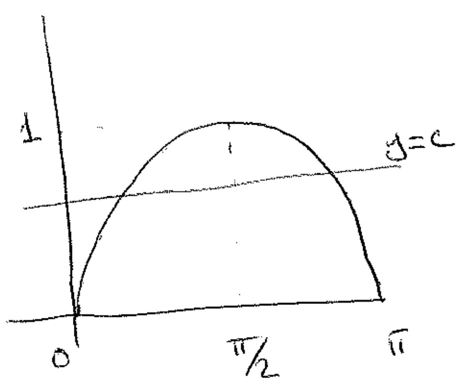
$$= \int_0^1 15\pi\,dy + \int_1^2 \pi(16-y^4)\,dy$$

$$= (15\pi y)\Big|_0^1 + \pi\left(16y - \frac{y^5}{5}\right)\Big|_1^2 = 15\pi + \frac{49}{5}\pi = \frac{124}{5}\pi.$$

6. The arch $y=\sin x$, $0 \le x \le \pi$ is revolved about the line $y=c$, $0 \le c \le 1$, to generate a solid S_c.

a) What is the value of $0 \le c \le 1$ that minimizes $V(S_c)$?

b) ______________________ maximizes ______?



$$V(S_c) = \int_0^\pi \pi R(x)^2\,dx$$

$$= \int_0^\pi \pi(\sin x - c)^2\,dx$$

(note that $(\sin x - c)^2 = (c - \sin x)^2$)

So $V(S_c) = \pi \int_0^\pi (\sin^2 x - 2c\sin x + c^2)\,dx$

$$= \pi\int_0^\pi \left(\frac{1-\cos 2x}{2} - 2c\sin x + c^2\right)dx$$

$$= \pi\left(\frac{1}{2}x - \frac{\sin 2x}{4} + 2c\cos x + c^2 x\right)\Big|_0^\pi$$

$$= \pi\left(\frac{\pi}{2} - \frac{\sin 2\pi}{4} + 2c\cos\pi + c^2\pi - 0 + \frac{\sin 0}{4} - 2c\cos 0 + c^2 0\right)$$

$$= \pi\left(\frac{\pi}{2} - 4c + \pi c^2\right)$$

To find min / max, we take the derivative with respect to c:

$$\left(\frac{\pi}{2} - 4c + \pi c^2\right)' = 2\pi c - 4$$

Set this to 0, get $c = 2/\pi$.

Now we check the values of $\pi/2 - 4c + \pi c^2$ at endpoints and at $c = 2/\pi$:

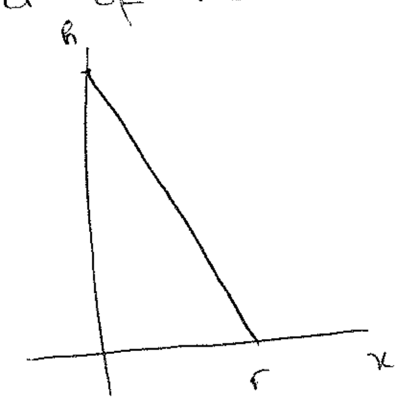
$c = 0$: $\pi/2 - 4c + \pi c^2 = \pi/2$.

$c = 2/\pi$: $\pi/2 - 4c + \pi c^2 = \frac{\pi}{2} - \frac{\cancel{8}4}{\pi}$

$c = 1$. $\pi/2 - 4c + \pi c^2 = \frac{3\pi}{2} - 4$

Note that $\frac{\pi}{2} > \frac{3\pi}{2} - 4 > \frac{\pi}{2} - \frac{4}{\pi}$, so we get the maximum at $c = 0$ and minimum at $c = 2/\pi$.

7. Derive the formula for the volume of a right circular cone of height h and radius r using an appropriate solid of revolution.



Revolve the ~~base~~ area bounded by $y = \frac{-h}{r}x + h$, $y = 0$, $x = 0$ about the y-axis.

$$V = \int_0^h \pi R(y)^2\, dy = \int_0^h \pi \left(\frac{-r}{h}(y-h) \right)^2 dy = \int_0^h \pi \left(\frac{-r}{h} y + r \right)^2 dy$$

$$= \int_0^h \pi \left(\frac{r^2}{h^2} y^2 - 2\frac{r^2}{h} y + r^2 \right) dy$$

$$= \pi \left(\frac{r^2}{h^2} \frac{y^3}{3} - \frac{r^2}{h} y^2 + r^2 y \right) \Big|_0^h$$

$$= \pi \left(\frac{r^2}{h^2} \frac{h^3}{3} - \frac{r^2}{h} h^2 + r^2 h \right) = \frac{\pi r^2 h}{3}$$