Homework 5 solutions

1. a) want to find c so that

$$\int_0^\infty cx^3e^{-x}\,dx = 1.$$

By integration by parts we get

$$\int x^3e^{-x} = -x^3e^{-x} + 3(-x^2e^{-x} + 2(-xe^{-x} - e^{-x}))$$

So
$$\int_0^\infty x^3e^{-x}dx = \lim_{a\to\infty}\int_0^a x^3e^{-x}dx$$

$$= \lim_{a\to\infty}(-a^3e^{-a} + 3(-a^2e^{-a} + 2(-ae^{-a} - e^{-a})) + 6)$$

$$\lim_{a\to\infty}\frac{a^3}{e^a} \overset{\text{l'Hopital}}{=} \lim_{a\to\infty}\frac{3a^2}{e^a} \overset{\text{l'H}}{=} \lim_{a\to\infty}\frac{6a}{e^a} \overset{\text{l'H}}{=} \lim_{a\to\infty}\frac{6}{e^a} = 0$$

Similarly, $\lim_{a\to\infty} a^2e^{-a} = 0$, $\lim_{a\to\infty} ae^{-a} = 0$, $\lim_{a\to\infty} e^{-a} = 0$

Therefore $\int_0^\infty x^3e^{-x}dx = 6$, i.e. $c = \frac{1}{6}$.

b) Mean $= \int_{-\infty}^\infty xf(x)\,dx$

$$= \frac{1}{6}\int_0^\infty x^4e^{-x}dx$$

$$= \frac{1}{6}\lim_{a\to\infty}(-a^4e^{-a} + 4(-a^3e^{-a} + 3(-a^2e^{-a} + 2(-ae^{-a} - e^{-a}))) + 24)$$

similar to part a $\;= \frac{1}{6}\cdot 24 = 4.$

2. $f(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{\mu} e^{-\frac{1}{\mu}x} & x \geq 0 \end{cases}$ (so that we get mean $= \mu$)

$$\int_0^\infty (x-\mu)^2 \frac{1}{\mu} e^{-\frac{1}{\mu}x}\, dx$$

$$= \lim_{a \to \infty} \int_0^a (x-\mu)^2 \frac{1}{\mu} e^{-\frac{1}{\mu}x}\, dx$$

$$= \lim_{a \to \infty} \left[-(x-\mu)^2 e^{-\frac{1}{\mu}x} + 2\left(-\mu(x-\mu)e^{-\frac{1}{\mu}x} - \mu^2 e^{-\frac{1}{\mu}x}\right) \right] \Big|_0^a$$

$$= \lim_{a \to \infty} \left[-(a-\mu)^2 e^{-\frac{1}{\mu}a} + 2\left(-\mu(a-\mu)e^{-\frac{1}{\mu}a} - \mu^2 e^{-\frac{1}{\mu}a}\right) + \mu^2 \right]$$

Similar to prob. 1, the part w/ a would go to 0 as $a \to \infty$, so $\int_0^\infty (x-\mu)^2 \frac{1}{\mu} e^{-\frac{1}{\mu}x} dx = \mu^2$

i.e. $\sigma = (\mu^2)^{1/2} = \mu.$

3. a) $\lim_{n\to\infty} \frac{1}{1+n^2} = 0$

b) $\lim_{n\to\infty} \frac{n}{n+3} = \lim_{n\to\infty} \frac{1}{1+3/n} = 1$

c) $\lim_{n\to\infty} \frac{1+n^2}{2+n} = \lim_{n\to\infty} \frac{1/n+n}{2/n+1} = \infty$

d) $\lim_{n\to\infty} 2^{1/n} = 2^0 = 1.$

e) $n! > n$ for all n, so a_n is unbounded, i.e. it diverges.

f) $a_n = \sqrt[n]{2n} = (2n)^{1/n}$

$\ln(a_n) = \frac{1}{n}\ln(2n)$

$\lim_{n\to\infty} \frac{\ln(2n)}{n} \overset{\text{l'Hopital}}{=} \lim_{n\to\infty} \frac{2\cdot\frac{1}{2n}}{1} = 0$

so $\lim_{n\to\infty} a_n = e^0 = 1$

(Note: Here we are implicitly using the fact that $\ln$ is continuous where it is defined)

g) $a_n = \frac{n!}{n^n} = \frac{1\cdot 2\cdot 3\cdots n}{\underbrace{n\cdot n\cdot n\cdots n}_{n \text{ times}}} = \frac{2\cdot 3\cdots n}{\underbrace{n\cdots n}_{n \text{ times}}}$

$= \frac{2}{n}\cdot\frac{3}{n}\cdots\frac{n}{n}\cdot\frac{1}{n} < 1\cdot 1\cdots 1\cdot\frac{1}{n} = \frac{1}{n}$

(for $n \geq 2$)

So we have $0 < a_n < \frac{1}{n}$ for all $n \geq 2$.

Since $\lim_{n\to\infty} \frac{1}{n} = 0$, by sandwich thm,

$$\lim_{n\to\infty} a_n = 0$$

b) $a_n = \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(\frac{n+1}{n}\right)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n}$

$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

So $\lim_{n\to\infty} a_n = \frac{1}{e}$.