Homework #8 Solutions

1. a) $f(x) = x^2 e^{3x}$, $x_0 = 0$

Solution: The taylor series for e^x is $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$. Hence

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \Rightarrow e^{3x} = \sum_{n=0}^{\infty} \frac{1}{n!} (3x)^n = \sum_{n=0}^{\infty} \frac{3^n}{n!} x^n \Rightarrow x^2 e^{3x} = \boxed{\sum_{n=0}^{\infty} \frac{3^n}{n!} x^{n+2}}$$

b) $f(x) = x\cos(x^2)$, $x_0 = 0$

Solution: The taylor series for $\cos(x)$ is $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$. Hence

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow \cos(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!} \Rightarrow$$

$$x\cos(x^2) = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(2n)!}}$$

c) $f(x) = \ln(1+2x+x^2)$

Solution: Note that $(1+2x+x^2) = (x+1)^2$. Hence $\ln(1+2x+x^2) =$

$= \ln((1+x)^2) = 2\ln(x+1)$.

As the taylor series for $\ln(x+1)$ is $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$, we thus have

$$\ln(1+2x+x^2) = \boxed{\sum_{n=1}^{\infty} 2(-1)^{n+1} \frac{x^n}{n}}.$$

2. Use the first three terms in the taylor series of $f(x) = x\cos(x^2)$ to approximate the value of $\cos(1/4)$.

Solution: By 1b), we have $x\cos(x^2) \sim x - \frac{x^5}{2} + \frac{x^9}{24}$.

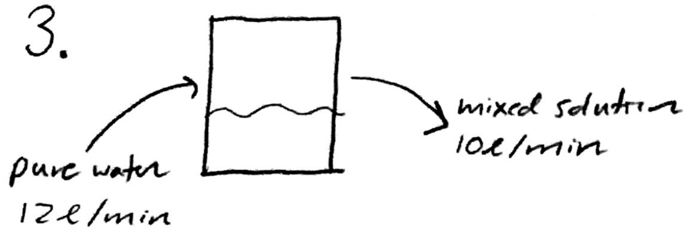
we have $\frac{1}{2}\cos((\frac{1}{2})^2) = \frac{1}{2}\cos(\frac{1}{4}) \Rightarrow 2\left(\frac{1}{2}\cos((\frac{1}{2})^2)\right) = \cos(1/4)$.

thus plugging in 1/2 to the above approximation, and multiplying by 2, gives an approximation for $\cos(1/4)$:

$$\frac{1}{2}\cos((\tfrac{1}{2})^2) \sim \frac{1}{2} - \frac{(1/2)^5}{2} + \frac{(1/2)^9}{24} = \frac{5953}{12288}.$$

Thus $\cos(\frac{1}{4}) \sim 2\left(\frac{5953}{12288}\right) = \boxed{\frac{5953}{6144}}$

3.



Initially: 10 kg salt dissolved in 100ℓ water.

How much salt will be dissolved in the water after 30mins?

Solution: Let $y(t)$ = salt in water at time t. Then:

$\frac{dy}{dt}$ = (rate salt in) − (rate salt out)

As pure water is flowing in, we have (rate salt in) = 0.

To find (rate salt out), note that $\frac{\text{salt}}{\ell} \cdot \frac{\ell}{\text{min}} = \text{salt}/\text{min}$.

We know ℓ/min out = 10. We have salt = $y(t)$. So we just need a formula for ℓ of water at time t. There are 100ℓ of water initially, and a net $12-10 = 2\,\ell/\text{min}$ of water rising in the tank.

Hence water at time $t = 100 + 2t$.

Thus (rate salt out) = $\frac{\text{salt}}{\ell} \cdot \frac{\ell}{\text{min}} = \frac{y}{100+2t} \cdot 10 = \frac{10y}{100+2t}$.

Thus our differential equation is:

$$\frac{dy}{dt} = -\frac{10y}{100+2t}.$$

This is seperable, so we have

$$-\frac{1}{10y}\,dy = \frac{1}{100+2t}\,dt \Rightarrow -\frac{1}{10}\ln(y) = \frac{1}{2}\ln(100+2t) + c$$

$$\Rightarrow \ln(y) = -5\ln(100+2t) + C \Rightarrow \ln(y) = \ln((100+2t)^{-5}) + C$$

$$\Rightarrow y = e^{\ln((100+2t)^{-5}) + C} = e^{C} \cdot e^{\ln((100+2t)^{-5})} = C \cdot (100+2t)^{-5}.$$

Using the initial condition: $y(0) = 10 \leadsto 10 = C(100)^{-5} \Rightarrow$

$C = 100000000000$. Thus $y(t) = \frac{100000000000}{(100+2t)^5}$, and

$$y(30) = \frac{100000000000}{(100+60)^5} = \boxed{\frac{15625}{16384} \sim .95 \text{ kg salt}}$$

4. a) $\frac{dy}{dt} = ry(t)\left(1 - \frac{y(t)}{K}\right)$

Solution: This is a seperable differential equation, so we have:

$\frac{1}{y(1-\frac{y}{K})} dy = r\,dt. = \int \frac{1}{y(1-\frac{y}{K})} dy = \int r\,dt$. For the LHS,

we do partial fractions:

$\frac{1}{y(1-\frac{y}{K})} = \frac{A}{y} + \frac{B}{1-\frac{y}{K}} \iff \frac{1}{y(1-\frac{y}{K})} = \frac{A(1-\frac{y}{K}) + By}{y(1-\frac{y}{K})}$

$\iff 1 = A + (B - \frac{A}{K})y \iff A = 1,\ B = 1/K$. Hence

$\int \frac{1}{y(1-y/K)} dy = \int \frac{1}{y} + \frac{1}{K(1-y/K)} dy = \int \frac{1}{y} + \frac{1}{K-y} dy$

$= \ln(y) - \ln(K-y) = \ln\left(\frac{y}{K-y}\right)$.

we have $\int r\,dt = rt + C$, so:

$\ln\left(\frac{y}{K-y}\right) = rt + C \Rightarrow Ce^{rt} = \frac{y}{K-y} \Rightarrow KCe^{rt} - Ce^{rt}y = y$

$\Rightarrow KCe^{rt} = (1 + Ce^{rt})y \Rightarrow y = \frac{KCe^{rt}}{1+Ce^{rt}}$.

To simplify, we can divide by Ce^{rt} to obtain:

$$\boxed{y = \frac{K}{Ce^{-rt}+1}}$$

b)

Solution: The exponential growth model is $y(t) = P_0 e^{rt}$.

we have $t=0$ in 1900 with population 76 million, so

$P_0 = 76$. Thus $y(t) = 76e^{rt}$. Since $y(50) = 150$,

we have $150 = 76e^{50r} \Rightarrow \ln\left(\frac{150}{76}\right) = 50r \Rightarrow r = \frac{1}{50}\ln\left(\frac{150}{76}\right)$

$\sim .0136$.

Hence $y(t) = 76e^{.0136t}$, and in 2000 the population

is $y(100) = 76e^{.0136 \cdot 100} \sim \boxed{296}$, and in 2050

$y(150) = 76e^{.0136 \cdot 150} \sim \boxed{584}$.

c) We now have enough data to find the constants K, C, and r in the solution to the logistic model.

We have:

$y(0)=76$: $\frac{K}{1+C}=76$

$y(50)=150$: $\frac{K}{Ce^{-r\cdot 50}+1}=150$

$y(100)=281$: $\frac{K}{Ce^{-r100}+1}=281.$

Solving this system of 3 equations in 3 unknowns gives: $K \approx 1421.07$, $C \approx 17.70$, $r \approx -.0147$.

Note that C is not itself the initial population.

Hence $y(t)=\frac{1421.07}{17.7e^{-.0147t}+1}$, and in 2050,

$y(150)=\frac{1421.07}{17.7e^{-.0147\cdot 150}+1} \approx \boxed{481}$

Note that this is a lower estimate than that obtained through the exponential growth model.