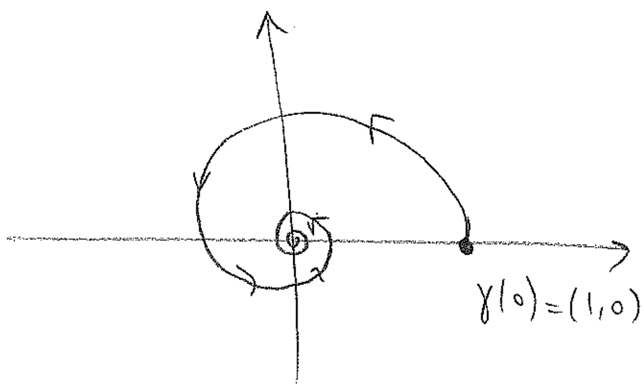
THE LOGARITHMIC SPIRAL

Bernoulli: "Eadem mutata resurgo"

$$\gamma(t) = \left(e^{-t}\cos t,\ e^{-t}\sin t\right)$$

<u>In nature:</u> Nautilus shell, vortices in fluids, cyclones, spiral galaxies...



By "Squeeze Thm", we have:

$$\lim_{t\to\infty} \gamma(t) = (0,0).$$

<u>Arc length parametrization:</u>

$$\gamma'(t) = \left(-e^{-t}\cos t - e^{-t}\sin t,\ -e^{-t}\sin + e^{-t}\cos t\right)$$

$$|\gamma'(t)|^2 = e^{-2t}\underbrace{\left(\cos t+\sin t\right)^2 + \left(\cos t - \sin t\right)^2}_{2} = 2e^{-2t}$$

$$\Rightarrow |\gamma'(t)| = \sqrt{2}\,e^{-t}$$

<u>Arc length:</u> $s(t) = \int_0^t \sqrt{2}\,e^{-\tau}\,d\tau = \sqrt{2}\left(-e^{-\tau}\right)\Big|_0^t = \sqrt{2}\left(1-e^{-t}\right).$

<u>Inverse function:</u> $t(s) = ?$

$$s = \sqrt{2}\left(1-e^{-t}\right) \Leftrightarrow \frac{s}{\sqrt{2}} - 1 = -e^{-t} \Leftrightarrow \ln\left(1-\frac{s}{\sqrt{2}}\right) = -t$$

$$\Leftrightarrow t = -\ln\left(1-\frac{s}{\sqrt{2}}\right) = \ln\left(\frac{1}{1-\frac{s}{\sqrt{2}}}\right).$$

Note: This means that the length of the logarithmic spiral from $(1,0)$ to $(0,0)$ is

$$\ell = \lim_{t \to \infty} s(t) = \lim_{t\to\infty} \sqrt{2}\left(1-e^{-t}\right) = \underline{\underline{\sqrt{2}}}.$$

Thus, the arclength parametrization of

$$\gamma(t) = \left(e^{-t}\cos t, e^{-t}\sin t\right), \quad 0 \leq t < \infty$$

is:

$$\gamma(s) = \left(e^{-t(s)}\cos t(s), e^{-t(s)}\sin t(s)\right)$$

$$\gamma(s) = \left(\left(1-\frac{s}{\sqrt{2}}\right)\cos\left[\ln\left(\frac{1}{1-\frac{s}{\sqrt{2}}}\right)\right], \left(1-\frac{s}{\sqrt{2}}\right)\sin\left[\ln\left(\frac{1}{1-\frac{s}{\sqrt{2}}}\right)\right]\right)$$

where $0 \leq s < \sqrt{2}$. Note: As $t \nearrow \infty$, $s \nearrow \sqrt{2}$.

Bonus Exercise: Show that the curvature of the logarithmic spiral is $\kappa(s) = |\gamma''(s)| = \dfrac{1}{\sqrt{2}-s}$

As expected, curvature blows up as the curve spirals down to the origin

In particular, observe that as $s \nearrow \sqrt{2}$:

$$\lim_{s\to\sqrt{2}_-} \kappa(s) = \lim_{s\to\sqrt{2}_-} \frac{1}{(\sqrt{2}-s)} = +\infty$$