Math 312, Spring 2016 **Renato Ghini Bettiol**

Homework Set 6

DUE: MAR 16, 2016 (IN CLASS)

1. Bretscher Section 5.4: 1, 4, 6, 7, 21, 31, 34

2. A *linear isometry* is a linear transformation $A\colon \mathbb{R}^n \to \mathbb{R}^n$ is that satisfies $\|Ax\| = \|x\|$ for all $x \in \mathbb{R}^n$.

   (a) Show that A is a linear isometry if and only if A is an orthogonal transformation, that is $\langle Ax, Ay\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{R}^n$.
   HINT: Use the *polarization identity* $\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$.

   (b) Show that A is an orthogonal transformation if and only if its matrix $A \in M_{n\times n}(\mathbb{R})$ is orthogonal (in symbols, $A \in \mathrm{O}(n)$), that is, $A^t A = \mathrm{Id}$.

3. A matrix $A \in M_{n\times n}(\mathbb{R})$ is called skew-symmetric if $A^t = -A$.

   (a) Give an example of a 2×2 skew-symmetric matrix.

   (b) Show that the diagonal entries of any skew-symmetric matrix are zero.

   (c) Show that if A is skew-symmetric, then $\langle Ax, x\rangle = 0$ for all $x \in \mathbb{R}^n$.

4. Let $\mathrm{Sym}_n \subset M_{n\times n}(\mathbb{R})$ be the subspace of symmetric $n \times n$ matrices and let $\mathrm{Skew}_n \subset M_{n\times n}(\mathbb{R})$ be the subspace of $n \times n$ skew-symmetric matrices.

   (a) Compute the dimension of Sym_n.

   (b) Compute the dimension of Skew_n.

   (c) Show that for all symmetric matrices $M, Q \in \mathrm{Sym}_n$ such that Q is invertible, there exists a matrix A satisfying the equation $AQ + QA^t = M$.
   HINT: Study the map $\Phi\colon M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ given by $\Phi(A) = AQ + QA^t$.