Chapter 1
Basic Results on Lie Groups

This chapter gives an introduction to Lie group theory, presenting the main concepts and giving detailed proofs of basic results. Some knowledge of group theory, linear algebra and advanced calculus is assumed. However, as a service to the reader, a few facts about differentiable manifolds are recalled in Appendix A, which can be used as preliminary reading.

The following references, which inspire our approach to the subject, can be used for further reading material on the contents of this chapter: Bump [58], Duistermaat and Kolk [79], Fegan [85], Gilmore [97], Gorbatsevich et al. [88, 98–100], Helgason [126], Hilgert and Neeb [128], Hsiang [129], Knapp [145], Onishchik [179], Spivak [198], Varadarajan [216] and Warner [227].

1.1 Lie Groups and Lie Algebras

Definition 1.1. A smooth (respectively, analytic) manifold G is said to be a *smooth* (respectively, *analytic*) *Lie group*[1] if G is a group and the maps

$$G \times G \ni (x,y) \longmapsto xy \in G \tag{1.1}$$

$$G \ni x \longmapsto x^{-1} \in G \tag{1.2}$$

are smooth (respectively, analytic).

[1]Sophus Lie was a nineteenth century Norwegian mathematician, who laid the foundations of continuous symmetry groups and transformation groups.

M.M. Alexandrino, R.G. Bettiol, *Lie Groups and Geometric Aspects of Isometric Actions*, DOI 10.1007/978-3-319-16613-1_1

Remark 1.2. The requirements that (1.1) and (1.2) be smooth, or analytic, often appear as a requirement that the map $G \times G \ni (x,y) \longmapsto xy^{-1} \in G$ be smooth, or analytic. It is easy to prove that these conditions are equivalent. See also Exercise 1.50.

In this book, we only deal with *smooth* Lie groups. Nevertheless, we mention the following important result, which is explored in detail in Duistermaat and Kolk [79].

Theorem 1.3. *Each C^2 Lie group admits a unique* analytic structure*, turning G into an analytic Lie group.*

We stress that every result proved in this chapter on (smooth) Lie groups is hence automatically valid on analytic Lie groups. Henceforth, G denotes a smooth Lie group, and the word *smooth* is omitted.

Remark 1.4. Historically, an important problem was to determine if a connected locally Euclidean topological group has a smooth structure. This problem was known as 5th *Hilbert's problem*, posed by Hilbert at the International Congress of Mathematicians in 1900, and solved by von Neumann in 1933 in the compact case. Only in 1952 the general case was solved, by Gleason, Montgomery and Zippen [168].

Some of the most basic examples of Lie groups are $(\mathbb{R}^n,+)$, $(S^1,\cdot)$ with group operation $e^{i\theta}\cdot e^{i\eta} = e^{i(\theta+\eta)}$, and the n-torus $T^n = S^1 \times \cdots \times S^1$ as a product group.

Example 1.5. More interesting examples are the so-called *classical Lie groups*, which form four families of matrix Lie groups, closely related to symmetries of Euclidean spaces. The term *classical* appeared in 1940 in Weyl's monograph, probably referring to the *classical* geometries in the spirit of Klein's Erlangen program.

We begin with $\mathrm{GL}(n,\mathbb{R})$, the *general linear group* of nonsingular (i.e., invertible) $n\times n$ real matrices. Similarly, $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{GL}(n,\mathbb{H})$ are respectively the groups of nonsingular $n \times n$ matrices over the complex numbers and the quaternions.[2] Furthermore, the following complete the list of classical Lie groups, where I denotes the identity matrix:

(i) $\mathrm{SL}(n,\mathbb{R}) = \{M \in \mathrm{GL}(n,\mathbb{R}) : \det M = 1\}$, $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{H})$, the *special linear groups*;
(ii) $\mathrm{O}(n) = \{M \in \mathrm{GL}(n,\mathbb{R}) : M^{\mathrm{t}}M = I\}$, the *orthogonal group*;
(iii) $\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}(n,\mathbb{R})$, the *special orthogonal group*;

[2]The *quaternions* form a real normed division algebra, which provides a noncommutative extension of complex (and real) numbers. Quaternions were discovered by Sir William Hamilton, after whom the usual notation is $\mathbb{H}$. The algebra $\mathbb{H}$ is a 4-dimensional real vector space, endowed with the quaternion multiplication. Its orthonormal basis is denoted $(1,i,j,k)$, and the product of basis elements is given by $i^2 = j^2 = k^2 = ijk = -1$. Any elements of $\mathbb{H}$ are of the form $a+b\,i+c\,j+d\,k \in \mathbb{H}$, and their product is determined by the equations above and the distributive law. Conjugation, norm and division can be defined as natural extensions of those in $\mathbb{C}$.

(iv) $\mathrm{U}(n) = \{M \in \mathrm{GL}(n,\mathbb{C}) : M^*M = I\}$, the *unitary group*;
(v) $\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}(n,\mathbb{C})$, the *special unitary group*;
(vi) $\mathrm{Sp}(n) = \{M \in \mathrm{GL}(n,\mathbb{H}) : M^*M = I\}$, the *symplectic group*.

In order to verify that those are indeed Lie groups, see Exercise 1.51. For now, we encourage the reader to keep them in mind as important examples of Lie groups.

Remark 1.6. Another class of examples of Lie groups is constructed by quotients of Lie groups by their normal and closed subgroups (see Corollary 3.38). In this class of examples, there are Lie groups that are *not* matrix groups. In fact, consider

$$G := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a,b,c \in \mathbb{R} \right\} \quad \text{and} \quad N := \left\{ \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}.$$

Then G/N is a Lie group. It is possible to prove that there are no injective homomorphisms $\varphi \colon G/N \to \mathrm{Aut}(V)$, where $\mathrm{Aut}(V)$ denotes the group of linear automorphisms of a finite-dimensional vector space V (see Carter, Segal and MacDonald [66]).

Definition 1.7. A *Lie algebra* $\mathfrak{g}$ is a real vector space endowed with a bilinear map $[\cdot,\cdot] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the *Lie bracket*, satisfying for all $X,Y,Z \in \mathfrak{g}$,

(i) Skew-symmetry: $[X,Y] = -[Y,X]$;
(ii) *Jacobi identity*: $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$.

Example 1.8. Basic examples of Lie algebras are the vector spaces of $n \times n$ square matrices over $\mathbb{R}$ and $\mathbb{C}$, respectively denoted $\mathfrak{gl}(n,\mathbb{R})$ and $\mathfrak{gl}(n,\mathbb{C})$, endowed with the Lie bracket given by the matrix commutator $[A,B] = AB - BA$.

Exercise 1.9. Let $\mathfrak{so}(3) = \{A \in \mathfrak{gl}(3,\mathbb{R}) : A^{\mathrm{t}} + A = 0\}$.

(i) Verify $\mathfrak{so}(3)$ is a Lie algebra, with Lie bracket given by the matrix commutator;
(ii) Let $A_X = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$. Prove that $A_X v = X \times v$.
(iii) Verify that $A_{X \times Y} = [A_X, A_Y] = A_X A_Y - A_Y A_X$. Using the fact that $(\mathbb{R}^3, \times)$ is a Lie algebra endowed with the cross product of vectors, conclude that the map $(\mathbb{R}^3, \times) \ni X \mapsto A_X \in \mathfrak{so}(3)$ is a *Lie algebra isomorphism*, i.e., a linear isomorphism that preserves Lie brackets.

We now proceed to associate to each Lie group a Lie algebra, by considering left-invariant vector fields. For each $g \in G$, denote by L_g and R_g the *left* and *right translation* maps on G, that is,

$$L_g(x) := gx \quad \text{and} \quad R_g(x) := xg. \tag{1.3}$$

These smooth maps are easily verified to be diffeomorphisms, since they have simultaneous left and right smooth inverses $L_{g^{-1}}$ and $R_{g^{-1}}$, respectively.

A (not necessarily smooth) vector field X on G is said to be *left-invariant* if X is L_g-related to itself for all $g \in G$, i.e., $\mathrm{d}L_g \circ X = X \circ L_g$. This means that $X(gh) = \mathrm{d}(L_g)_h X(h)$, or shortly $X = \mathrm{d}L_g X$, for all $g \in G$. Similarly, a vector field is *right-invariant* if it is R_g-related to itself for all $g \in G$, meaning that $X = \mathrm{d}R_g X$ for all $g \in G$. A simultaneously left- and right-invariant vector field is said to be *bi-invariant*.

Lemma 1.10. *Left-invariant vector fields are smooth.*

Proof. Let X be a left-invariant vector field on G, and consider the group operation

$$\mu\colon G\times G \to G, \quad \mu(g,h) = gh.$$

Differentiating μ, we obtain $\mathrm{d}\mu\colon T(G\times G) \cong TG\times TG \to TG$, a smooth map. Define $s\colon G \to TG\times TG$ by $s(g) := (0_g, X(e))$, where $g \mapsto 0_g$ is the null section of TG. Since $X = \mathrm{d}\mu \circ s$, it follows that X is smooth. □

Remark 1.11. Clearly, the above result also holds for *right-invariant* vector fields.

Given two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, a linear map $\psi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is called *Lie algebra homomorphism* if

$$[\psi(X), \psi(Y)] = \psi([X,Y]), \quad \text{for all } X, Y \in \mathfrak{g}_1.$$

Theorem 1.12. *Let $\mathfrak{g}$ be the set of left-invariant vector fields on the Lie group G. Then the following hold:*

(i) *$\mathfrak{g}$ is a Lie algebra, endowed with the Lie bracket of vector fields;*
(ii) *Consider the tangent space T_eG with the bracket defined as follows. If $X^1, X^2 \in T_eG$, set $[X^1, X^2] := [\widetilde{X^1}, \widetilde{X^2}]_e$ where $\widetilde{X^i_g} = \mathrm{d}(L_g)_e X^i$. Define $\psi\colon \mathfrak{g} \to G$ by $\psi(X) := X_e$. Then ψ is a Lie algebra isomorphism, where $\mathfrak{g}$ is endowed with the Lie bracket of vector fields and T_eG with the bracket defined above.*

Proof. First, note that $\mathfrak{g}$ has a (real) vector space structure, by the linearity of $\mathrm{d}(L_g)_e$. It is not difficult to see that the Lie bracket of vector fields is a Lie bracket, i.e., it is skew-symmetric and satisfies the Jacobi identity. Equation (A.3) implies that the Lie bracket of left-invariant vector fields is still left-invariant. Hence $\mathfrak{g}$ is a Lie algebra, proving (i).

To prove (ii), we first claim that ψ is injective. Indeed, if $\psi(X) = \psi(Y)$, for each $g \in G$, $X(g) = \mathrm{d}L_g(X(e)) = \mathrm{d}L_g(Y(e)) = Y(g)$. Furthermore, it is also surjective, since for each $v \in T_eG$, $X(g) := \mathrm{d}L_g(v)$ is clearly left-invariant and satisfies $\psi(X) = v$. Therefore, ψ is a linear bijection between two vector spaces, hence an isomorphism. From the definition of Lie bracket on T_eG, we have $[\psi(X), \psi(Y)] = [X,Y]_e = \psi([X,Y])$. Thus ψ is a Lie algebra isomorphism. □

Definition 1.13. The *Lie algebra of the Lie group G* is the Lie algebra $\mathfrak{g}$ of left-invariant vector fields on G.

According to the above theorem, $\mathfrak{g}$ could be equivalently defined as the tangent space T_eG, with the bracket defined as in (ii). In this way, a Lie group G gives rise to a canonically determined Lie algebra $\mathfrak{g}$. A converse is given by the following foundational result, see e.g., Duistermaat and Kolk [79].

Lie's Third Theorem 1.14. *Let $\mathfrak{g}$ be a (finite-dimensional) Lie algebra. There exists a unique connected and simply-connected Lie group G with Lie algebra isomorphic to $\mathfrak{g}$.*

We do not prove the existence of G, but its uniqueness follows from Corollary 1.27 below. We conclude this section with an exercise that complements Exercise 1.9, verifying that $\mathfrak{so}(3)$ is the Lie algebra of $\mathrm{SO}(3)$.

Exercise 1.15. Assume the following result to be seen in Exercise 1.38: *The tangent space at the identity to a Lie subgroup of* $\mathrm{GL}(n,\mathbb{R})$*, endowed with the matrix commutator, is isomorphic to its Lie algebra.* Consider $\mathscr{S} \subset \mathrm{GL}(3,\mathbb{R})$ the subset of symmetric matrices, and define $\varphi\colon \mathrm{GL}(3,\mathbb{R}) \to \mathscr{S}$, $\varphi(A) = AA^{\mathrm{t}}$.

(i) Verify that the kernel of $\mathrm{d}\varphi(I)\colon \mathfrak{gl}(3,\mathbb{R}) \to \mathscr{S}$ is the subspace of skew-symmetric matrices in $\mathfrak{gl}(3,\mathbb{R})$;
(ii) Prove that, for all $A \in \mathrm{GL}(3,\mathbb{R})$,

$$\ker \mathrm{d}\varphi(A) = \left\{ H \in \mathfrak{gl}(3,\mathbb{R}) : A^{-1}H \in \ker \mathrm{d}\varphi(I) \right\}.$$

Conclude that $\dim \ker \mathrm{d}\varphi(A) = 3$, for all $A \in \mathrm{GL}(3,\mathbb{R})$;
(iii) Prove that the identity matrix I is a regular value of φ;
(iv) Conclude that the Lie group $\mathrm{O}(3)$ may be written as $\varphi^{-1}(I)$, and compute its dimension.
(v) Recall that $\mathrm{SO}(3)$ is the subgroup of $\mathrm{O}(3)$ determined by the connected component of the identity I. Prove that $T_I\mathrm{SO}(3) = \ker \mathrm{d}\varphi(I) = \mathfrak{so}(3)$. Finally, conclude that $\mathfrak{so}(3)$ is the Lie algebra of $\mathrm{SO}(3)$.

Analogous statements hold for $\mathrm{SO}(n)$ and $\mathfrak{so}(n)$, with identical proofs.

1.2 Lie Subgroups and Lie Homomorphisms

The aim of this section is to establish the basic relations between Lie algebras and Lie groups, together with their natural maps and subobjects.

A group homomorphism between Lie groups $\varphi\colon G_1 \to G_2$ is called a *Lie group homomorphism* if it is smooth. In Corollary 1.49, we prove that continuity is actually sufficient for a group homomorphism between Lie groups to be smooth. Recall that, given two Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$, a linear map $\psi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is a *Lie algebra* *homomorphism* if $\psi([X,Y]) = [\psi(X), \psi(Y)]$, for all $X, Y \in \mathfrak{g}_1$. A *Lie subgroup H* of a Lie group G is an abstract subgroup, such that H is an immersed submanifold of G and

$$H \times H \ni (x,y) \longmapsto xy^{-1} \in H \tag{1.4}$$

is smooth. In addition, if $\mathfrak{g}$ is a Lie algebra, a subspace $\mathfrak{h}$ is a *Lie subalgebra* if it is closed with respect to the Lie bracket.

Proposition 1.16. *Let G be a Lie group and $H \subset G$ an embedded submanifold of G that is also a group with respect to the group operation of G. Then H is a closed Lie subgroup of G.*

Proof. Consider the map $f\colon H \times H \to G$ given by (1.4). Then f is smooth and $f(H \times H) \subset H$. Since H is embedded in G, from Proposition A.7, $f\colon H \times H \to H$ is smooth. Hence H is a Lie subgroup of G.

It remains to prove that H is a closed subgroup of G. Since H is an embedded submanifold, there exists a neighborhood W of the identity $e \in G$, and a submanifold chart $\varphi = (x_1, \dots, x_k)\colon W \to \mathbb{R}^{\dim G}$, such that

$$H \cap W = \big\{ g \in G \cap W : x_i(g) = 0,\ i = 1, \dots \dim H \big\}. \tag{1.5}$$

Consider a sequence $\{h_n\}$ in H that converges to $h \in G$. Continuity of (1.4) implies the existence of a neighborhood U of $e \in G$ such that $UU^{-1} \subset W$. As h_n converges to h, for n sufficiently large, $h_n h^{-1} \in U$. In particular, $h_n h_m^{-1} \in H \cap W$ for n sufficiently large. From (1.5), this means that $x_i(h_n h_m^{-1}) = 0$ for $i = 1, \dots, \dim H$. Fixing m and letting $n \to +\infty$, we have that $x_i(h h_m^{-1}) = 0$ for $i = 1, \dots, \dim H$. Therefore, $h h_m^{-1} \in H \cap W$, hence $h \in H$, concluding the proof. □

A related result to Proposition 1.16 is proved in Sect. 1.4. We now investigate the nature of the Lie algebra of a Lie subgroup.

Lemma 1.17. *Let G_1 and G_2 be Lie groups and $\varphi\colon G_1 \to G_2$ be a Lie group homomorphism. Given any left-invariant vector field $X \in \mathfrak{g}_1$, there exists a unique left-invariant vector field $Y \in \mathfrak{g}_2$ that is φ-related to X.*

Proof. First, if $Y \in \mathfrak{g}_2$ is φ-related to X, since φ is a Lie group homomorphism, $Y_e = \mathrm{d}\varphi_e X_e$. Hence, uniqueness follows from left-invariance of Y.

Define $Y := \mathrm{d}(L_g)_e(\mathrm{d}\varphi_e(X_e))$. It remains to prove that Y is φ-related to X. Observing that φ is a Lie group homomorphism, $\varphi \circ L_g = L_{\varphi(g)} \circ \varphi$, for all $g \in G_1$. Therefore, for each $g \in G_1$,

$$\begin{aligned}
\mathrm{d}\varphi_g(X_g) &= \mathrm{d}\varphi_g(\mathrm{d}(L_g)_e X_e) \\
&= \mathrm{d}(\varphi \circ L_g)_e X_e \\
&= \mathrm{d}(L_{\varphi(g)} \circ \varphi)_e X_e
\end{aligned}$$

$$= \mathrm{d}(L_{\varphi(g)})_e(\mathrm{d}\varphi_e X_e)$$
$$= Y(\varphi(g)).$$

This shows that Y is φ-related to X, completing the proof. □

Proposition 1.18. *Let G_1 and G_2 be Lie groups and $\varphi\colon G_1 \to G_2$ be a Lie group homomorphism. Then $\mathrm{d}\varphi_e\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebra homomorphism.*

Proof. We want to prove that $\mathrm{d}\varphi_e\colon (T_eG_1, [\cdot,\cdot]) \to (T_eG_2, [\cdot,\cdot])$ is a Lie algebra homomorphism, where the Lie bracket $[\cdot,\cdot]$ on T_eG_i was defined in Theorem 1.12. For each $X^1, X^2 \in T_eG_1$, define the vectors $Y^i := \mathrm{d}\varphi_e X^i \in T_eG_2$ and extend them to left-invariant vector fields $\widetilde{X}^i \in \mathfrak{X}(G_1)$ and $\widetilde{Y}^i \in \mathfrak{X}(G_2)$, by setting $\widetilde{X}^i_g := \mathrm{d}L_g X^i$ and $\widetilde{Y}^i_g := \mathrm{d}L_g Y^i$.

On the one hand, it follows from the definition of the Lie bracket in T_eG_2 that

$$[\mathrm{d}\varphi_e X^1, \mathrm{d}\varphi_e X^2] = [Y^1, Y^2] = [\widetilde{Y}^1, \widetilde{Y}^2]_e.$$

On the other hand, it follows from Lemma 1.17 that $\widetilde{X}^i$ and $\widetilde{Y}^i$ are φ-related, and from (A.3), $[\widetilde{X}^1, \widetilde{X}^2]$ and $[\widetilde{Y}^1, \widetilde{Y}^2]$ are φ-related. Therefore

$$[\widetilde{Y}^1, \widetilde{Y}^2]_e = \mathrm{d}\varphi_e [\widetilde{X}^1, \widetilde{X}^2]_e = \mathrm{d}\varphi_e [X^1, X^2].$$

The above imply that $[\mathrm{d}\varphi_e X^1, \mathrm{d}\varphi_e X^2] = \mathrm{d}\varphi_e [X^1, X^2]$, concluding the proof. □

Corollary 1.19. *Let G be a Lie group and $H \subset G$ a Lie subgroup. Then the inclusion map $i\colon H \hookrightarrow G$ induces an isomorphism $\mathrm{d}i_e$ between the Lie algebra $\mathfrak{h}$ of H and a Lie subalgebra $\mathrm{d}i_e(\mathfrak{h})$ of $\mathfrak{g}$.*

A converse result is given below, on conditions under which a Lie subalgebra gives rise to a Lie subgroup. Before that, however, we need a result on how an open neighborhood of the identity *generates* the entire Lie group.

Proposition 1.20. *Let G be a Lie group and G_0 be the connected component of G containing the identity $e \in G$. Then G_0 is a normal Lie subgroup of G and connected components of G are of the form gG_0, for some $g \in G$. Moreover, for any open neighborhood U of e, $G_0 = \bigcup_{n\in\mathbb{N}} U^n$, where $U^n := \{g_1^{\pm 1} \cdots g_n^{\pm 1} : g_i \in U\}$.*

Proof. Since G_0 is a connected component, it is an open and closed subset of G. In order to verify that it is also a Lie subgroup, let $g_0 \in G_0$ and consider $g_0 G_0 = L_{g_0}(G_0)$. Note that $g_0 G_0$ is a connected component of G, as L_{g_0} is a diffeomorphism. Since $g_0 \in G_0 \cap g_0 G_0$, it follows from the maximality of the connected component that $g_0 G_0 = G_0$. Similarly, as the inversion map is also a diffeomorphism, the subset $G_0^{-1} := \{g_0^{-1} : g_0 \in G_0\}$ is connected, with $e \in G_0^{-1}$. Hence, $G_0^{-1} = G_0$, using the same argument. Therefore, G_0 is a subgroup of G and an embedded submanifold of G. From Proposition 1.16, it follows that G_0 is a Lie subgroup of G.

For each $g \in G$, consider the diffeomorphism given by conjugation $x \mapsto g x g^{-1}$. Using the same argument of maximality of the connected component, one may conclude that $g G_0 g^{-1} = G_0$ for all $g \in G$, hence G_0 is normal. The proof that $g G_0$ is the connected component of G containing g is completely analogous.

Finally, since G_0 connected, to show that $G_0 = \bigcup_{n\in\mathbb{N}} U^n$ it suffices to check that $\bigcup_{n\in\mathbb{N}} U^n$ is open and closed in G_0. It is clearly open, since U (hence U^n) is open. To verify that it is also closed, let $h \in G_0$ be the limit of a sequence $\{h_j\}$ in $\bigcup_{n\in\mathbb{N}} U^n$, i.e., $\lim h_j = h$. Since $U^{-1} = \{u^{-1} : u \in U\}$ is an open neighborhood of $e \in G$, hU^{-1} is an open neighborhood of h. From the convergence of the sequence $\{h_j\}$, there exists $j_0 \in \mathbb{N}$ such that $h_{j_0} \in hU^{-1}$, that is, there exists $u \in U$ such that $h_{j_0} = hu^{-1}$. Hence $h = h_{j_0} u \in \bigcup_{n\in\mathbb{N}} U^n$. Therefore, this set is closed in G_0. □

Theorem 1.21. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, and $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. There exists a unique connected Lie subgroup $H \subset G$ with Lie algebra $\mathfrak{h}$.*

Proof. Define the distribution $D_q = \{X_q : X_q = \mathrm{d}L_q X \text{ for } X \in \mathfrak{h}\}$. Since $\mathfrak{h}$ is a Lie algebra, D is involutive. It follows from the Frobenius Theorem (see Theorem A.19) that there exists a unique foliation $\mathscr{F} = \{F_q : q \in G\}$ tangent to this distribution, i.e., $D_q = T_q F_q$, for all $q \in G$. Define $H := F_e$ to be the leaf passing through the identity.

Note that L_g maps leaves to leaves, since $\mathrm{d}L_g D_a = D_{ga}$. Furthermore, for each $h \in H$, $L_{h^{-1}}(H)$ is the leaf passing through the identity. Therefore $L_{h^{-1}}(H) = H$, which means that H is a group. It also follows from the Frobenius Theorem that H is quasi-embedded. Consider $\psi \colon H \times H \ni (x,y) \mapsto x^{-1}y \in H$. Since the inclusion $i \colon H \hookrightarrow G$ and $i \circ \psi$ are smooth, ψ is also smooth. Thus, H is a connected Lie subgroup of G with Lie algebra $\mathfrak{h}$. Uniqueness follows from the Frobenius Theorem and Proposition 1.20. □

Definition 1.22. A smooth surjective map $\pi \colon E \to B$ is a *covering map* if, for each $p \in B$, there exists an open neighborhood U of p such that $\pi^{-1}(U)$ is a disjoint union of open sets $U_\alpha \subset E$ mapped diffeomorphically onto U by π. In other words, $\pi|_{U_\alpha} \colon U_\alpha \to U$ is a diffeomorphism for each α.

Theorem 1.23. *Let G be a connected Lie group. There exist a unique simply-connected Lie group $\widetilde{G}$ and a Lie group homomorphism $\pi \colon \widetilde{G} \to G$, which is also a covering map.*

A proof of this theorem can be found in Boothby [46] or Duistermaat and Kolk [79]. An example of covering map that is also a Lie group homomorphism is the usual covering $\pi \colon \mathbb{R}^n \to T^n$ of the n-torus by Euclidean space. Other examples of Lie group coverings are discussed in Exercise 1.55 and Remark 1.56.

Proposition 1.24. *Let G_1 and G_2 be connected Lie groups and $\pi \colon G_1 \to G_2$ be a Lie group homomorphism. Then π is a covering map if and only if $\mathrm{d}\pi_e$ is an isomorphism.*

Proof. Suppose that $\mathrm{d}\pi_{e_1} \colon T_{e_1} G_1 \to T_{e_2} G_2$ is an isomorphism, where $e_i \in G_i$ denotes the identity element of G_i. We claim that π is surjective.

Indeed, since $\mathrm{d}\pi_{e_1}$ is an isomorphism, from the Inverse Function Theorem, there exist open neighborhoods U of the identity $e_1 \in G_1$ and V of the identity $e_2 \in G_2$, such that $\pi(U) = V$ and $\pi|_U$ is a diffeomorphism. Let $h \in G_2$. From Proposition 1.20, there exist $h_i \in V$ such that $h = h_1^{\pm 1} \cdots h_n^{\pm 1}$. Since $\pi|_U$ is a diffeomorphism, for each $1 \leq i \leq n$, there exists a unique $g_i \in U$ such that $\pi(g_i^{\pm 1}) = h_i^{\pm 1}$. Thus,

$$\pi(g_1^{\pm 1} \cdots g_n^{\pm 1}) = \pi(g_1^{\pm 1}) \cdots \pi(g_n^{\pm 1}) = h_1^{\pm 1} \cdots h_n^{\pm 1} = h.$$

Therefore π is a surjective Lie group homomorphism.

Let $\{g_\alpha\} = \pi^{-1}(e_2)$. Using the fact that π is a Lie group homomorphism and $\mathrm{d}\pi_{e_1}$ is an isomorphism, one can prove that $\{g_\alpha\}$ is discrete. Thus, π is a covering map, since:

(i) $\pi^{-1}(qV) = \bigcup_\alpha g_\alpha pU$;
(ii) $(g_\alpha pU) \cap (pU) = \emptyset$, if $g_\alpha \neq e_1$;
(iii) $\pi\big|_{g_\alpha pU} \colon g_\alpha pU \to qV$ is a diffeomorphism.

Conversely, if π is a covering map, it is locally a diffeomorphism, hence $\mathrm{d}\pi_{e_1}$ is an isomorphism. □

Let G_1 and G_2 be Lie groups and $\theta \colon \mathfrak{g}_1 \to \mathfrak{g}_2$ be a Lie algebra homomorphism. We prove that if G_1 is connected and simply-connected, then θ induces a Lie group homomorphism. We begin by proving uniqueness in the following lemma.

Lemma 1.25. *Let G_1 and G_2 be Lie groups, with identities e_1 and e_2 respectively, and $\theta \colon \mathfrak{g}_1 \to \mathfrak{g}_2$ be a fixed Lie algebra homomorphism. If G_1 is connected, and $\varphi, \psi \colon G_1 \to G_2$ are Lie group homomorphisms with $\mathrm{d}\varphi_{e_1} = \mathrm{d}\psi_{e_1} = \theta$, then $\varphi = \psi$.*

Proof. It is easy to see that the direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ of Lie algebras has a natural Lie algebra structure, and the direct product $G_1 \times G_2$ of Lie groups has a natural Lie group structure. Consider the Lie subalgebra of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ given by

$$\mathfrak{h} := \big\{ (X, \theta(X)) : X \in \mathfrak{g}_1 \big\}, \tag{1.6}$$

i.e., the graph of $\theta \colon \mathfrak{g}_1 \to \mathfrak{g}_2$. It follows from Theorem 1.21 that there exists a unique connected Lie subgroup $H \subset G_1 \times G_2$ with Lie algebra $\mathfrak{h}$.

Suppose that $\varphi \colon G_1 \to G_2$ is a Lie group homomorphism with $\mathrm{d}\varphi_{e_1} = \theta$. Then

$$\sigma \colon G_1 \to G_1 \times G_2, \quad \sigma(g) := \big(g, \varphi(g)\big),$$

is a Lie group homomorphism, and

$$\mathrm{d}\sigma_{e_1} \colon \mathfrak{g}_1 \to \mathfrak{g}_1 \oplus \mathfrak{g}_2, \quad \mathrm{d}\sigma_{e_1} X = \big(X, \theta(X)\big)$$

is a Lie algebra homomorphism. Note that $\sigma(G_1)$ is the graph of φ, hence embedded in $G_1 \times G_2$. From Proposition 1.16, $\sigma(G_1)$ is a Lie subgroup of $G_1 \times G_2$, with Lie algebra $\mathfrak{h} = \mathrm{d}\sigma_{e_1}(\mathfrak{g}_1)$. Therefore, from Theorem 1.21, $\sigma(G_1) = H$. In other words, H is the graph of φ. If $\psi\colon G_1 \to G_2$ is another Lie group homomorphism with $\mathrm{d}\psi_{e_1} = \theta$, following the same construction above, the graphs of ψ and φ would both be equal to H, hence $\varphi = \psi$. □

Theorem 1.26. *Let G_1 and G_2 be Lie groups and $\theta\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ be a Lie algebra homomorphism. There exist an open neighborhood V of e_1 and a smooth map $\varphi\colon V \to G_2$ that is a local homomorphism*[3] *with $\mathrm{d}\varphi_{e_1} = \theta$. In addition, if G_1 is connected and simply-connected, there exists a unique Lie group homomorphism $\varphi\colon G_1 \to G_2$ with $\mathrm{d}\varphi_{e_1} = \theta$.*

Proof. Let $\mathfrak{h}$ be defined as in (1.6). From Theorem 1.21, there exists a unique connected Lie subgroup $H \subset G_1 \times G_2$ with Lie algebra $\mathfrak{h}$, whose identity element we denote $\widetilde{e} \in H$. Consider the inclusion $i\colon H \hookrightarrow G_1 \times G_2$ and the projections $\pi_j\colon G_1 \times G_2 \to G_j$, $j = 1,2$. The map $(\pi_1 \circ i)\colon H \to G_1$ is a Lie group homomorphism, such that $\mathrm{d}(\pi_1 \circ i)_{\widetilde{e}}(X, \theta(X)) = X$, for all $X \in T_{e_1}G_1$. It follows from the Inverse Function Theorem that there exist open neighborhoods U of $\widetilde{e}$ in H and V of e_1 in G_1, such that $(\pi_1 \circ i)|_U\colon U \to V$ is a diffeomorphism.

Define $\varphi = \pi_2 \circ (\pi_1 \circ i)^{-1}\colon V \to G_2$. Then φ is a local homomorphism with $\mathrm{d}\varphi_{e_1} = \theta$. In fact, for each $X \in T_{e_1}G_1$,

$$\begin{aligned}
\mathrm{d}\varphi_{e_1}(X) &= \mathrm{d}\big(\pi_2 \circ (\pi_1 \circ i)^{-1}\big)_{e_1}(X) \\
&= \mathrm{d}(\pi_2)_{\widetilde{e}}\, \mathrm{d}\big((\pi_1 \circ i)^{-1}\big)_{e_1} X \\
&= \mathrm{d}(\pi_2)_{\widetilde{e}}(X, \theta(X)) \\
&= \theta(X).
\end{aligned}$$

Furthermore, $(\pi_1 \circ i)$ is a Lie group homomorphism and $\mathrm{d}(\pi_1 \circ i)_{\widetilde{e}}$ is an isomorphism. From Proposition 1.24, $(\pi_1 \circ i)\colon H \to G_1$ is a covering map. Assuming G_1 simply-connected, it follows that $(\pi_1 \circ i)$ is a diffeomorphism, since a covering map onto a simply-connected space is a diffeomorphism. Thus, $(\pi_1 \circ i)$ can be globally inverted, and we obtain a global homomorphism $\varphi = \pi_2 \circ (\pi_1 \circ i)^{-1}\colon G_1 \to G_2$ with $\mathrm{d}\varphi_{e_1} = \theta$. The uniqueness of φ follows from Lemma 1.25, under the assumption that G_1 is simply-connected. □

Corollary 1.27. *If G_1 and G_2 are connected and simply-connected and $\theta\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is an isomorphism, there exists a unique Lie group isomorphism $\varphi\colon G_1 \to G_2$ with $\mathrm{d}\varphi_{e_1} = \theta$. In other words, if G_1 and G_2 are as above, then G_1 and G_2 are isomorphic if and only if $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic.*

Proof. By Theorem 1.26, there exists a unique Lie group homomorphism $\varphi\colon G_1 \to G_2$ with $\mathrm{d}\varphi_{e_1} = \theta$. By Proposition 1.24, since $\mathrm{d}\varphi_{e_1} = \theta$ is an isomorphism, φ is a covering map. Since G_2 is simply-connected, φ is a diffeomorphism. Thus, φ is a Lie group homomorphism and a diffeomorphism, therefore an isomorphism. □

[3] This simply means that $\varphi(ab) = \varphi(a)\varphi(b)$, for all $a, b \in V$ such that $ab \in V$.

1.3 Exponential Map and Adjoint Representation

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. We recall that a Lie group homomorphism $\varphi\colon \mathbb{R} \to G$ is called a 1-*parameter subgroup* of G. Let $X \in \mathfrak{g}$ and consider the Lie algebra homomorphism

$$\theta\colon \mathbb{R} \to \mathfrak{g}, \quad \theta(t) := tX.$$

From Theorems 1.21 and 1.26, there is a unique 1-parameter subgroup $\lambda_X\colon \mathbb{R} \to G$, such that $\lambda_X'(0) = X$.

Remark 1.28. Note that λ_X is an integral curve of the left-invariant vector field X passing through e. In fact,

$$\lambda_X'(t) = \tfrac{\mathrm{d}}{\mathrm{d}s}\lambda_X(t+s)\big|_{s=0} = \mathrm{d}L_{\lambda_X(t)}\lambda_X'(0) = \mathrm{d}L_{\lambda_X(t)}X = X(\lambda_X(t)).$$

Definition 1.29. The *(Lie) exponential map* of G is defined as

$$\exp\colon \mathfrak{g} \to G, \quad \exp(X) := \lambda_X(1),$$

where λ_X is the unique 1-parameter subgroup of G such that $\lambda_X'(0) = X$.

Proposition 1.30. *The exponential map satisfies the following properties, for all* $X \in \mathfrak{g}$ *and* $t \in \mathbb{R}$*,*

(i) $\exp(tX) = \lambda_X(t)$*;*
(ii) $\exp(-tX) = \exp(tX)^{-1}$*;*
(iii) $\exp(t_1X + t_2X) = \exp(t_1X)\exp(t_2X)$*;*
(iv) $\exp\colon T_eG \to G$ *is smooth and* $\mathrm{d}(\exp)_0 = \mathrm{id}$*, hence* $\exp$ *is a diffeomorphism from an open neighborhood of the origin of* T_eG *onto an open neighborhood of the identity* $e \in G$*.*

Proof. We claim that $\lambda_X(t) = \lambda_{tX}(1)$. Consider the 1-parameter subgroup $\lambda(s) = \lambda_X(st)$. Differentiating at $s = 0$, it follows that

$$\lambda'(0) = \tfrac{\mathrm{d}}{\mathrm{d}s}\lambda_X(st)\big|_{s=0} = t\lambda_X'(0) = tX.$$

Hence, from uniqueness of the 1-parameter subgroup in Definition 1.29, $\lambda_X(st) = \lambda_{tX}(s)$. Choosing $s = 1$, we obtain the expression in (i). Items (ii) and (iii) are immediate consequences of (i), since λ_X is a Lie group homomorphism.

In order to prove item (iv), we construct a vector field V on the tangent bundle TG. This tangent bundle can be identified with $G \times T_eG$, since G is parallelizable. The construction is such that the projection of the integral curve of V passing through (e,X) coincides with the curve $t \mapsto \exp(tX)$. Thus, from Theorem A.14, the flow of V depends smoothly on the initial conditions, hence its projection (i.e., the exponential map) is also smooth.

Consider $G\times T_eG\simeq TG$. Note that for all $(g,X)\in G\times T_eG$, the tangent space $T_{(g,X)}(G\times T_eG)$ can be identified with $T_gG\oplus T_eG$. Define the vector field

$$V\in\mathfrak{X}(G\times T_eG),\quad V(g,X):=\widetilde{X}(g)\oplus\{0\}\in T_gG\oplus T_eG,$$

where $\widetilde{X}(g)=\mathrm{d}L_g X$. It is not difficult to see that V is a smooth vector field. Since $t\mapsto\exp(tX)$ is the unique integral curve of $\widetilde{X}$ for which $\lambda_X(0)=e$, as $\widetilde{X}$ is left-invariant, $L_g\circ\lambda_X$ is the unique integral curve of $\widetilde{X}$ that takes value g at $t=0$. Hence, the integral curve of V through (g,X) is $t\mapsto(g\exp(tX),X)$. In other words, the flow of V is given by $\varphi_t^V(g,X)=(g\exp(tX),X)$ and, in particular, V is complete. Let $\pi_1\colon G\times T_eG\to G$ be the projection onto G. Then $\exp(X)=\pi_1\circ\varphi_1^V(e,X)$, so exp is given by composition of smooth maps and is hence smooth. Finally, (i) and Remark 1.28 imply that $\mathrm{d}(\exp)_0=\mathrm{id}$. □

The exponential map, in general, may not be surjective. The classical example of this situation is given by $\mathrm{SL}(2,\mathbb{R})$, see Duistermaat and Kolk [79].

Considering Lie groups of matrices $\mathrm{GL}(n,\mathbb{K})$, for $\mathbb{K}=\mathbb{C}$ or $\mathbb{K}=\mathbb{R}$, it is natural to inquire whether the Lie exponential map $\exp\colon\mathfrak{gl}(n,\mathbb{K})\to\mathrm{GL}(n,\mathbb{K})$ coincides with the usual exponentiation of matrices, given for each $A\in\mathfrak{gl}(n,\mathbb{K})$ by

$$e^A:=\sum_{k=0}^{\infty}\frac{A^k}{k!}.\tag{1.7}$$

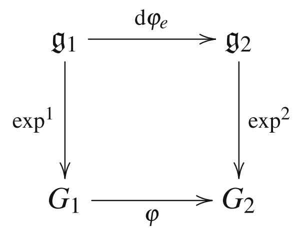
We now prove that these notions indeed coincide.

To this aim, we recall two well-known properties of the exponentiation of matrices. First, the right-hand side of (1.7) converges uniformly for A in a bounded region of $\mathfrak{gl}(n,\mathbb{K})$. This can be easily verified using the Weierstrass M-test. In addition, given $A,B\in\mathrm{GL}(n,\mathbb{K})$, it is true that $e^{A+B}=e^Ae^B$ if and only if A and B commute (this fact is generalized in Remark 1.40).

Consider the map $\mathbb{R}\ni t\mapsto e^{tA}\in\mathrm{GL}(n,\mathbb{K})$. Since each entry of e^{tA} is a power series in t with infinite radius of convergence, it follows that this map is smooth. Differentiating the power series term by term, it is easy to see that its tangent vector at the origin of $\mathfrak{gl}(n,\mathbb{K})$ is A, and from the properties above, this map is also a Lie group homomorphism, hence a 1-parameter subgroup of $\mathrm{GL}(n,\mathbb{K})$. Since $\exp(A)$ is the *unique* 1-parameter subgroup of $\mathrm{GL}(n,\mathbb{K})$ whose tangent vector at the origin is A, it follows that $e^A=\exp(A)$, for all $A\in\mathfrak{gl}(n,\mathbb{K})$.

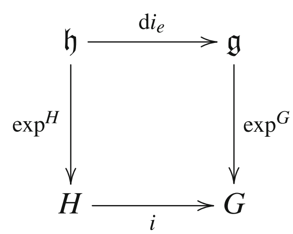
Remark 1.31. The above holds, more generally, for the exponential of endomorphisms of any real or complex vector space V. Namely, $\exp\colon\mathrm{End}(V)\to\mathrm{Aut}(V)$ is given by the exponentiation of endomorphisms, defined exactly as in (1.7).

Proposition 1.32. *Let G_1 and G_2 be Lie groups and $\varphi\colon G_1\to G_2$ a Lie group homomorphism. Then $\varphi\circ\exp^1=\exp^2\circ\mathrm{d}\varphi_e$, i.e., the following diagram commutes:*



Proof. Consider the 1-parameter subgroups of G_2 given by $\alpha(t) = \varphi \circ \exp^1(tX)$ and $\beta(t) = \exp^2 \circ \mathrm{d}\varphi_e(tX)$. Then $\alpha'(0) = \beta'(0) = \mathrm{d}\varphi_e X$, hence, it follows from Theorem 1.26 that $\alpha = \beta$, that is, the diagram above commutes. $\square$

Remark 1.33. Using Proposition 1.32, we now show that if H is a Lie subgroup of G, then the exponential map $\exp^H$ of H coincides with the restriction to H of the exponential map $\exp^G$ of G. Consider the inclusion $i\colon H \hookrightarrow G$, which is a Lie group homomorphism, and its differential $\mathrm{d}i_e\colon \mathfrak{h} \hookrightarrow \mathfrak{g}$. According to Corollary 1.19, this is an isomorphism between the Lie algebra $\mathfrak{h}$ and a Lie subalgebra of $\mathfrak{g}$. Then the following diagram is commutative



Thus, with the appropriate identifications, $\exp^H(X) = i(\exp^H(X)) = \exp^G(\mathrm{d}i_e(X)) = \exp^G(X)$, for all $X \in \mathfrak{h}$, which proves the assertion.

We proceed by stating three important identities known as the *Campbell formulas*, or *Campbell-Baker-Hausdorff formulas*. A proof can be found in Spivak [198].

Campbell formulas 1.34. *Let G be a Lie group and $X, Y \in \mathfrak{g}$. There exists $\varepsilon > 0$ such that, for all $|t| < \varepsilon$, the following hold:*

(i) $\exp(tX)\exp(tY) = \exp\left(t(X+Y) + \frac{t^2}{2}[X,Y] + O(t^3)\right)$;
(ii) $\exp(tX)\exp(tY)\exp(-tX) = \exp\left(tY + t^2[X,Y] + O(t^3)\right)$;
(iii) $\exp(-tX)\exp(-tY)\exp(tX)\exp(tY) = \exp\left(t^2[X,Y] + O(t^3)\right)$;

where $\frac{O(t^3)}{t^3}$ is bounded.

We now pass to the second part of this section, discussing some properties of the adjoint representation, or adjoint action.

Definition 1.35. Let G be any group and V a vector space. A *(linear) representation* of G on V is a group homomorphism $\varphi\colon G \to \mathrm{Aut}(V)$, where $\mathrm{Aut}(V)$ is the group of vector space isomorphisms of V.

Consider the action of G on itself by conjugation, i.e.,

$$a\colon G\times G\to G,\quad a(g,x):=a_g(x)=gxg^{-1}.$$

Definition 1.36. Let G be a Lie group, and $\mathfrak{g}$ its Lie algebra. The representation

$$\mathrm{Ad}\colon G\to\mathrm{Aut}(\mathfrak{g}),\quad \mathrm{Ad}(g):=\mathrm{d}(a_g)_e=\mathrm{d}(L_g)_{g^{-1}}\circ\mathrm{d}(R_{g^{-1}})_e,$$

is called the *adjoint representation* (or *adjoint action*) of G.

It follows from the above definition that

$$\mathrm{Ad}(g)X=\tfrac{\mathrm{d}}{\mathrm{d}t}\left(g\exp(tX)g^{-1}\right)\big|_{t=0}. \tag{1.8}$$

Applying Proposition 1.32 to the automorphism a_g, it follows that

$$\exp(t\mathrm{Ad}(g)X)=a_g(\exp(tX))=g\exp(tX)g^{-1}, \tag{1.9}$$

and, in particular, for $t=1$,

$$g\exp(X)g^{-1}=\exp(\mathrm{Ad}(g)X). \tag{1.10}$$

The differential of the adjoint representation Ad is denoted ad, and given by

$$\mathrm{ad}\colon\mathfrak{g}\to\mathrm{End}(\mathfrak{g}),\quad \mathrm{ad}(X)Y:=\mathrm{d}\mathrm{Ad}_e(X)(Y).$$

It follows from the above definition that

$$\mathrm{ad}(X)Y=\tfrac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{Ad}\left(\exp(tX)\right)Y\right)\big|_{t=0}. \tag{1.11}$$

Using Proposition 1.32 once more, we have

$$\mathrm{Ad}(\exp(tX))=\exp(t\mathrm{ad}(X))$$

i.e., the following diagram is commutative:

$$\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\ \mathrm{ad}\ } & \mathrm{End}(\mathfrak{g})\\
{\scriptstyle\exp}\downarrow & & \downarrow{\scriptstyle\exp}\\
G & \xrightarrow[\ \mathrm{Ad}\ ]{} & \mathrm{Aut}(\mathfrak{g})
\end{array}$$

In particular, for $t=1$ we obtain

$$\mathrm{Ad}(\exp(X))=\exp(\mathrm{ad}(X)) \tag{1.12}$$

rbettiol@math.upenn.edu

Proposition 1.37. *If* $X,Y \in \mathfrak{g}$, *then* $\mathrm{ad}(X)Y = [X,Y]$.

Proof. From the Campbell formulas (Theorem 1.34), it follows that

$$\exp(tX)\exp(tY)\exp(-tX) = \exp\left(tY + t^2[X,Y] + O(t^3)\right).$$

Using (1.10) with $g = \exp(tX)$, $\exp\left(\mathrm{Ad}(\exp(tX))tY\right) = \exp(tY + t^2[X,Y] + O(t^3))$. Since exp is locally injective near zero, we have $\mathrm{Ad}(\exp(tX))tY = tY + t^2[X,Y] + O(t^3)$, for sufficiently small t. Dividing by t, differentiating at $t = 0$, and applying (1.11), we conclude that $\mathrm{ad}(X)Y = [X,Y]$. □

Exercise 1.38. Let G be a Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$. For each $g \in G$ and $X,Y \in \mathfrak{g}$, verify the following properties:

(i) $\mathrm{d}L_g X = gX$ and $\mathrm{d}R_g X = Xg$ are given by matrix multiplication;
(ii) $\mathrm{Ad}(g)Y = gYg^{-1}$ is given by matrix conjugation;
(iii) Use Proposition 1.37 to prove that $[X,Y] = XY - YX$ is the matrix commutator.

Let us now prove a result relating commutativity and the Lie bracket. Recall that vector fields $X,Y \in \mathfrak{X}(M)$ are said to *commute* if $[X,Y] = 0$.

Proposition 1.39. *Let* G *be a connected Lie group with Lie algebra* $\mathfrak{g}$. *The Lie algebra* $\mathfrak{g}$ *is abelian if and only if* G *is abelian.*

Proof. Fix $X,Y \in \mathfrak{g}$ and assume that $[X,Y] = 0$. From (1.12) and Proposition 1.37,

$$\mathrm{Ad}(\exp(X))Y = \exp(\mathrm{ad}(X))Y = \sum_{k=0}^{\infty} \frac{\mathrm{ad}(X)^k}{k!} Y = Y.$$

Thus, from (1.10), $\exp(X)\exp(Y)\exp(-X) = \exp(Y)$, and hence $\exp(X)\exp(Y) = \exp(Y)\exp(X)$. This means that there exists an open neighborhood U of e such that if $g_1, g_2 \in U$, then $g_1 g_2 = g_2 g_1$. It follows from Proposition 1.20, that $G = \bigcup_{n\in\mathbb{N}} U^n$, where $U^n = \{g_1^{\pm 1} \cdots g_n^{\pm 1} : g_i \in U\}$. Therefore G is abelian.[4]

Conversely, suppose G abelian. In particular, for all $s,t \in \mathbb{R}$ and $X,Y \in \mathfrak{g}$, $\exp(sX)\exp(tY)\exp(-sX) = \exp(tY)$. Differentiating at $t = 0$,

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \exp(sX)\exp(tY)\exp(-sX)\big|_{t=0} = Y.$$

From (1.8), it follows that $\mathrm{Ad}(\exp(sX))Y = Y$. Hence, differentiating at $s = 0$, it follows from (1.11) and Proposition 1.37 that $\mathrm{ad}(X)Y = [X,Y] = 0$, which means that $\mathfrak{g}$ is abelian. □

Remark 1.40. If $X,Y \in \mathfrak{g}$ commute, that is, $[X,Y] = 0$, then

$$\exp(X+Y) = \exp(X)\exp(Y).$$

[4] Note that one cannot infer that G is abelian directly from the commutativity of exp, since exp might not be surjective.

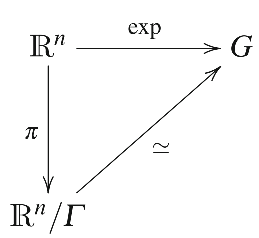
Note that this does not hold in general. To verify this identity, set $\alpha\colon \mathbb{R}\to G$, $\alpha(t) := \exp(tX)\exp(tY)$. From Proposition 1.39, α is a 1-parameter subgroup, and differentiating α at $t=0$, we have $\alpha'(0)=X+Y$. Hence $\alpha(t)=\exp(t(X+Y))$, and we get the desired equation setting $t=1$.

We conclude this section with a characterization of connected abelian Lie groups.

Theorem 1.41. *Let G be a connected n-dimensional abelian Lie group. Then G is isomorphic to $T^k\times\mathbb{R}^{n-k}$, where $T^k=S^1\times\cdots\times S^1$ is a k-torus. In particular, an abelian connected and compact Lie group is isomorphic to a torus.*

Proof. Using Proposition 1.39, since G is abelian, $\mathfrak{g}$ is also abelian. Thus $\mathfrak{g}$ can be identified with $\mathbb{R}^n$. Since G is connected, it follows from Remark 1.40 that $\exp\colon \mathfrak{g}\to G$ is a Lie group homomorphism. From Proposition 1.24, it is a covering map.

Consider the normal subgroup given by $\Gamma=\ker\exp$. We must now use two results whose proof only appears later. The first (Theorem 1.42) asserts that any closed subgroup of a Lie group is a Lie subgroup. As exp continuous and Γ is closed, Γ is a Lie subgroup of $\mathbb{R}^n$. The second (Corollary 3.38) asserts that the quotient of a Lie group by a normal Lie subgroup is also a Lie group, hence $\mathbb{R}^n/\Gamma$ is a Lie group. Note that G is isomorphic to $\mathbb{R}^n/\Gamma$, because $\exp\colon \mathbb{R}^n\to G$ is a surjective Lie group homomorphism and $\Gamma=\ker\exp$.



Using the fact that exp is a covering map, it is possible to prove that the isomorphism between $\mathbb{R}^n/\Gamma$ and G defined above is in fact smooth, i.e., is a Lie group isomorphism (this also follows from Corollary 1.49).

Finally, it is well-known that the only nontrivial discrete subgroups of $\mathbb{R}^n$ are integral lattices. In other words, there exists a positive integer $k\leq n$ and linearly independent vectors $e_1,\dots,e_k\in\mathbb{R}^n$ such that $\Gamma=\left\{\sum_{i=1}^k n_ie_i : n_i\in\mathbb{Z}\right\}$. Therefore, G is isomorphic to $\mathbb{R}^n/\Gamma = T^k\times\mathbb{R}^{n-k}$, where $T^k=S^1\times\cdots\times S^1$ is a k-torus. □

1.4 Closed Subgroups and More Examples

The goal of this section is to prove that closed subgroups of a Lie group are Lie subgroups. This fact is a very useful tool to prove that a subgroup is a *Lie subgroup*. For instance, it applies to the subgroups of $\mathrm{GL}(n,\mathbb{K})$, for $\mathbb{K}=\mathbb{C}$ and $\mathbb{K}=\mathbb{R}$, defined in Sect. 1.1. We then briefly explore some corollaries, and discuss a few important examples that complement the material of this chapter.

Theorem 1.42. *Let G be a Lie group and $H \subset G$ a closed subgroup of G. Then H is an embedded Lie subgroup of G.*

Proof. We prove this result through a sequence of five claims, following closely the approach in Spivak [198, pp. 530–534]. The central idea of the proof is to reconstruct the Lie algebra of H as a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The natural candidate is

$$\mathfrak{h} := \big\{X \in T_eG : \exp(tX) \in H, \text{ for all } t \in \mathbb{R}\big\}.$$

Claim 1.43. Let $\{X_i\}$ be a sequence in T_eG with $\lim X_i = X$, and $\{t_i\}$ a sequence of real numbers, with $\lim t_i = 0$. If $\exp(t_iX_i) \in H$, for all $i \in \mathbb{N}$, then $X \in \mathfrak{h}$.

Since $\exp(-t_iX_i) = \big(\exp(t_iX_i)\big)^{-1}$, without loss of generality, assume $t_i > 0$. Define $R_i(t)$ to be the largest integer $\leq \frac{t}{t_i}$. Then $\frac{t}{t_i} - 1 < R_i(t) \leq \frac{t}{t_i}$, so $\lim t_iR_i(t) = t$. Therefore $\lim t_iR_i(t)X_i = tX$. On the one hand, it follows from continuity of exp that $\lim \exp(t_iR_i(t)X_i) = \exp(tX)$. On the other hand, $\exp(t_iR_i(t)X_i) = [\exp(t_iX_i)]^{R_i(t)} \in H$. Since H is closed, $\exp(tX) \in H$. Therefore $X \in \mathfrak{h}$.

Claim 1.44. $\mathfrak{h} \subset T_eG$ is a vector subspace of $\mathfrak{g}$.

Let $X, Y \in \mathfrak{h}$. It is clear that $sX \in \mathfrak{h}$ for all $s \in \mathbb{R}$. From the Campbell formulas (Theorem 1.34),

$$\exp\left(t_i(X+Y) + \tfrac{t_i^2}{2}[X,Y] + O(t_i^3)\right) = \exp(t_iX)\exp(t_iY) \in H,$$

so $\exp\big(t_i(X+Y+\frac{t_i}{2}[X,Y]+O(t_i^2))\big) \in H$ and $\big(X+Y+\frac{t_i}{2}[X,Y]+O(t_i^2)\big)$ converges to $X+Y$ when t_i goes to 0. From Claim 1.43, $X+Y \in \mathfrak{h}$.

Claim 1.45. Let $\mathfrak{k}$ be a vector space such that $T_eG = \mathfrak{h} \oplus \mathfrak{k}$, and

$$\psi\colon \mathfrak{h} \oplus \mathfrak{k} \to G, \quad \psi(X,Y) := \exp(X)\exp(Y).$$

There exists an open neighborhood U of the origin $(0,0) \in \mathfrak{h} \oplus \mathfrak{k}$, such that $\psi|_U$ is a diffeomorphism.

Differentiating ψ with respect to each component, it follows that

$$\mathrm{d}\psi_{(0,0)}(X,0) = \mathrm{d}(\exp)_0X = X,$$
$$\mathrm{d}\psi_{(0,0)}(0,Y) = \mathrm{d}(\exp)_0Y = Y.$$

Thus, $\mathrm{d}\psi_0 = \mathrm{id}$. From the Inverse Function Theorem, there exists U an open neighborhood of the origin, such that $\psi|_U$ is a diffeomorphism.

Claim 1.46. There exists an open neighborhood V of the origin of $\mathfrak{k}$, such that $\exp(Y) \notin H$ for $Y \in V \setminus \{0\}$.

Suppose that there exists a sequence $\{Y_i\}$ with $Y_i \in \mathfrak{k}$, $\lim Y_i = 0$ and $\exp(Y_i) \in H$. Choose an inner product on $\mathfrak{k}$ and define $t_i = \|Y_i\|$ and $X_i = \frac{1}{t_i} Y_i$. Since $\{X_i\}$ is a sequence in the unit sphere of $\mathfrak{k}$, which is compact, it converges up to passing to a subsequence. Hence $\lim X_i = X$, $\lim t_i = 0$ and $\exp(t_i X_i) \in H$. Therefore, from Claim 1.43, $X \in \mathfrak{h}$. This contradicts $\mathfrak{h} \cap \mathfrak{k} = \{0\}$.

Claim 1.47. There exists an open neighborhood W of the origin in T_eG such that $H \cap \exp(W) = \exp(\mathfrak{h} \cap W)$.

It follows from the construction of $\mathfrak{h}$ that $H \cap \exp(W) \supset \exp(\mathfrak{h} \cap W)$. According to Claims 1.45 and 1.46, there exists a sufficiently small open neighborhood W of the origin of T_eG such that $\exp|_W$ and $\psi|_W$ are diffeomorphisms, and $(W \cap \mathfrak{k}) \subset V$.

Let $a \in H \cap \exp(W)$. As $\psi|_W$ is a diffeomorphism, there exist a unique $X \in \mathfrak{h}$ and a unique $Y \in \mathfrak{k}$ such that $a = \exp(X)\exp(Y)$. Hence $\exp(Y) = (\exp(X))^{-1} a \in H$. From Claim 1.46, $Y = 0$, that is, $a = \exp(X)$, with $X \in \mathfrak{h}$. Therefore $H \cap \exp(W) \subset \exp(\mathfrak{h} \cap W)$. From Claim 1.47, H is an embedded submanifold of G in a neighborhood of the identity $e \in G$. Thus, since H is a group, it is an embedded submanifold. Finally, from Proposition 1.16, H is an embedded Lie subgroup of G. □

In order to explore some consequences of the above result, we recall the *Constant Rank Theorem*. It states that if a smooth map $f\colon M \to N$ is such that $\mathrm{d}f_x$ has constant rank, then for each $x_0 \in M$ there exists a neighborhood U of x_0 such that:

(i) $f(U)$ is an embedded submanifold of N;
(ii) The partition $\{f^{-1}(y) \cap U\}_{y \in f(U)}$ is a foliation of U;
(iii) For each $y \in f(U)$, $\ker \mathrm{d}f_x = T_x\big(f^{-1}(y)\big)$, for all $x \in f^{-1}(y)$.

Lemma 1.48. *Let G_1 and G_2 be Lie groups and $\varphi\colon G_1 \to G_2$ be a Lie group homomorphism. Then the following hold:*

(i) $\mathrm{d}\varphi_g$ *has constant rank;*
(ii) $\ker \varphi$ *is a normal Lie subgroup of* G_1*;*
(iii) $\ker \mathrm{d}\varphi_e = T_e(\ker \varphi)$.

Proof. Since φ is a Lie group homomorphism, $\varphi \circ L^1_g = L^2_{\varphi(g)} \circ \varphi$, where L^i_g denotes the left multiplication by g on G_i. Hence, for all $X \in T_gG_1$,

$$\mathrm{d}\varphi_g X = \mathrm{d}\varphi_g\, \mathrm{d}(L^1_g)_e\, \mathrm{d}(L^1_{g^{-1}})_g X = \mathrm{d}L^2_{\varphi(g)}\, \mathrm{d}\varphi_e\, \mathrm{d}(L^1_{g^{-1}})_g X.$$

Since $L^2_{\varphi(g)}$ is a diffeomorphism, $\mathrm{d}\varphi_g X = 0$ if and only if $\mathrm{d}\varphi_e \mathrm{d}(L^1_{g^{-1}})X = 0$. Hence $\dim\ker \mathrm{d}\varphi_g = \dim\ker \mathrm{d}\varphi_e$, therefore $\mathrm{d}\varphi_g$ has constant rank, proving (i). Item (ii) follows from Theorem 1.42, since $\ker\varphi = \varphi^{-1}(e)$ is a closed normal subgroup. Finally, (iii) follows directly from the Constant Rank Theorem. □

Corollary 1.49. *Let G_1 and G_2 be Lie groups and $\varphi\colon G_1 \to G_2$ a continuous homomorphism. Then φ is smooth.*

Proof. Let $R = \{(g, \varphi(g)) : g \in G_1\}$ be the graph of φ. Then R is a closed subgroup of $G_1 \times G_2$, hence, from Theorem 1.42, R is an embedded Lie subgroup of $G_1 \times G_2$. Consider $i\colon R \hookrightarrow G_1 \times G_2$ the inclusion and the projections $\pi_j\colon G \times G \to G_j$, $j = 1,2$. Then $(\pi_1 \circ i)$ is a Lie group homomorphism, and from Lemma 1.48, $\mathrm{d}(\pi_1 \circ i)_g$ has constant rank. On the other hand, R is a graph, hence, by the Constant Rank Theorem, $(\pi_1 \circ i)$ is an immersion.

In addition, $\dim R = \dim G_1$, otherwise $(\pi_1 \circ i)(R)$ would have measure zero, contradicting $(\pi_1 \circ i)(R) = G_1$. From the Inverse Function Theorem, $(\pi_1 \circ i)$ is a local diffeomorphism. Since it is also bijective, it is a global diffeomorphism, therefore $\varphi = \pi_2 \circ (\pi_1 \circ i)^{-1}$ is smooth. □

Exercise 1.50 ($\star$). Let G be a Lie group and denote by $\mu\colon G \times G \to G$ the multiplication map $\mu(g,h) = gh$.

(i) Show that the differential $\mathrm{d}\mu_{(g,h)}\colon T_gG \oplus T_hG \to T_{gh}G$ is given by

$$\mathrm{d}\mu_{(g,h)}(X,Y) = \mathrm{d}L_g Y + \mathrm{d}R_h X, \quad X \in T_gG, Y \in T_hG, \tag{1.13}$$

and conclude that $\mu\colon G \times G \to G$ is a submersion (see Definition A.4);

(ii) Show that $\mu^{-1}(e) = \{(g, g^{-1}) : g \in G\}$ is a submanifold of $G \times G$, and describe its tangent space $T_{(g,h)}\big(\mu^{-1}(e)\big) \subset T_gG \oplus T_hG$;

(iii) Denote by $i\colon \mu^{-1}(e) \hookrightarrow G \times G$ the inclusion and by $\pi_j\colon G \times G \to G$, $j = 1,2$, the projection maps. Show that the restriction $\pi_1\colon \mu^{-1}(e) \to G$ is a diffeomorphism. Conclude that smoothness of the inversion map (1.2), which can be written as $\pi_2 \circ (\pi_1 \circ i)^{-1}\colon G \to G$, is a consequence of the smoothness of μ.

Exercise 1.51. Prove that $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{U}(n)$, $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$ are Lie groups (recall definitions in Example 1.5). Verify that their Lie algebras are, respectively,

(i) $\mathfrak{gl}(n,\mathbb{R})$, the space of $n \times n$ square matrices over $\mathbb{R}$;
(ii) $\mathfrak{gl}(n,\mathbb{C})$, the space of $n \times n$ square matrices over $\mathbb{C}$;
(iii) $\mathfrak{sl}(n,\mathbb{R}) = \{X \in \mathfrak{gl}(n,\mathbb{R}) : \operatorname{tr} X = 0\}$;
(iv) $\mathfrak{sl}(n,\mathbb{C}) = \{X \in \mathfrak{gl}(n,\mathbb{C}) : \operatorname{tr} X = 0\}$;
(v) $\mathfrak{o}(n) = \mathfrak{so}(n) = \{X \in \mathfrak{gl}(n,\mathbb{R}) : X^{\mathrm{t}} + X = 0\}$;
(vi) $\mathfrak{u}(n) = \{X \in \mathfrak{gl}(n,\mathbb{C}) : X^* + X = 0\}$;
(vii) $\mathfrak{su}(n) = \mathfrak{u}(n) \cap \mathfrak{sl}(n,\mathbb{C})$;
(viii) $\mathfrak{sp}(n) = \{X \in \mathfrak{gl}(n,\mathbb{H}) : X^* + X = 0\}$.

Compare (v) with Exercise 1.15.

Hint: Prove directly that $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{GL}(n,\mathbb{C})$ are Lie subgroups and verify that the others are closed subgroups of these. Use Remark 1.33 to prove that if $X \in \mathfrak{g}$ and $\mathfrak{h} \subset \mathfrak{g}$, then $X \in \mathfrak{h}$ if and only if $\exp(tX) \in H$, for all $t \in \mathbb{R}$. Then, use Lemma 1.48, combined with the identity $\det e^A = e^{\operatorname{tr} A}$ for any $A \in \mathfrak{gl}(n,\mathbb{K})$, to assist with the computation of the above Lie algebras.

Exercise 1.52. Let $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ be the unit 3-sphere. Verify that S^3 is a Lie group when endowed with the multiplication

$$(z_1, z_2) \cdot (w_1, w_2) := \big(z_1 w_1 - \overline{z_2} w_2,\ w_1 z_2 + \overline{z_1} w_2\big).$$

(i) Consider the map $\varphi \colon S^3 \to \mathrm{SU}(2)$, given by

$$\varphi(z_1, z_2) = \begin{pmatrix} z_1 & -\overline{z_2} \\ z_2 & \overline{z_1} \end{pmatrix}$$

Verify that φ is a continuous homomorphism, hence a Lie group homomorphism. Verify that $\ker\varphi$ is trivial, so φ is an isomorphism;

(ii) Consider the group of unit quaternions

$$\mathrm{Sp}(1) = \big\{a + b\,i + c\,j + d\,k \in \mathbb{H} : a^2 + b^2 + c^2 + d^2 = 1\big\}$$

Find a continuous (hence smooth) group homomorphism $\psi \colon \mathrm{Sp}(1) \to S^3$;

(iii) Conclude that S^3, $\mathrm{Sp}(1)$ and $\mathrm{SU}(2)$ are isomorphic Lie groups.

Exercise 1.53. Let G be a Lie group and consider $\rho : \widetilde{G} \to G$ its universal covering. Prove that:

(i) $H := \rho^{-1}(e)$ is a normal discrete closed subgroup, and $g\,h = h\,g$, for all $h \in H, g \in \widetilde{G}$;

(ii) $G \cong \widetilde{G}/H$;

(iii) $\pi_1(G)$ is abelian.

Hint: Item (i) follows directly from ρ being a continuous homomorphism. The fact that $g\,h = h\,g$ can be proved defining $f \colon \widetilde{G} \to H$ as $f(x) := x\,h\,x^{-1}$, noticing that $\{h\}$ is an open and closed subset of H and hence concluding that $f^{-1}(\{h\})$ is an open and closed subset of $\widetilde{G}$. In order to prove (iii), consider two loops α_i in G such that $\alpha_i(0) = e$, and their lifts $\widetilde{\alpha}_i$ with $\widetilde{\alpha}_i(0) = e$. Set $h_i = \widetilde{\alpha}_i(1)$ and use (i) to conclude that $\widetilde{\alpha_2 * \alpha_1} = h_2\,h_1 = h_1\,h_2 = \widetilde{\alpha_1 * \alpha_2}$ where $*$ denotes concatenation of curves.

Further details on the smooth structure of quotients of Lie groups, such as $\widetilde{G}/H$ above, are given in Chap. 3, see Corollary 3.38.

Remark 1.54. The above exercise gives a glimpse of the structure of homotopy groups of Lie groups. By using Morse theory on the space of paths on a compact Lie group G, it is possible to prove that, in addition to $\pi_1(G)$ being abelian, $\pi_2(G)$ is trivial and $\pi_3(G)$ is torsion-free. These techniques also led to the discovery that homotopy groups of classical Lie groups are periodic, which is a foundational result known as *Bott periodicity*. For further details, we refer the reader to Milnor [158].

Exercise 1.55 ($\star$)**.** Prove that $\mathrm{SU}(2)$ is the universal covering of $\mathrm{SO}(3)$, via the following steps:

(i) Let $g \in \mathrm{SU}(2) \cong S^3$, $u \in S^2 \subset \mathbb{R}^3$ and $\theta \in [0, 2\pi]$ be such that $g = \cos(\theta) + \sin(\theta)u$. Define $T_g(v) = g v g^{-1}$ for all $v \in \mathbb{R}^3$. Prove that $T_g \colon \mathbb{R}^3 \to \mathbb{R}^3$ is a linear orthogonal transformation, and $T_g = e^{A_{2\theta u}}$, where A_X is as in Exercise 1.9;
(ii) Prove that $\varphi \colon S^3 \ni g \mapsto T_g \in \mathrm{SO}(3)$ is a double covering map, concluding $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$.

Hint: Recall that S^3 is isomorphic to $\mathrm{SU}(2)$, see Exercise 1.52. Also, recall that the product of quaternions corresponds to $(t_1 + u_1) \cdot (t_2 + u_2) = (t_1 t_2 - \langle u_1, u_2 \rangle) + (t_2 u_1 + t_1 u_2 + u_1 \times u_2)$, and verify that $\frac{\mathrm{d}}{\mathrm{d}t} T_{\cos(t\theta)+\sin(t\theta)u}(v)|_{t=0} = 2\theta\, u \times v$. Note that under the identifications $\mathrm{SU}(2) \cong S^3$ and $\mathrm{SO}(3) \cong \mathbb{R}P^3$, the double covering $\mathrm{SU}(2) \to \mathrm{SO}(3)$ is precisely the double covering $S^3 \to \mathbb{R}P^3$.

Remark 1.56. Using results from basic topology, it is not difficult to show that $\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}_2$ for all $n \geq 3$. The universal covering of $\mathrm{SO}(n)$, $n \geq 3$, is called the *spin group* and denoted $\mathrm{Spin}(n)$, recall Theorem 1.23. Algebraic manipulations slightly more elaborate than those in Exercise 1.55 show that $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$ is a double covering of $\mathrm{SO}(4)$, $\mathrm{Sp}(2)$ is a double covering of $\mathrm{SO}(5)$, and $\mathrm{SU}(4)$ is a double covering of $\mathrm{SO}(6)$. In particular, we have the isomorphisms:

$$\mathrm{Spin}(3) \cong \mathrm{Sp}(1), \;\; \mathrm{Spin}(4) \cong \mathrm{Sp}(1) \times \mathrm{Sp}(1), \;\; \mathrm{Spin}(5) \cong \mathrm{Sp}(2), \; \mathrm{Spin}(6) \cong \mathrm{SU}(4).$$

The *center* of a Lie group G is the subgroup given by

$$Z(G) := \left\{ g \in G : g h g^{-1} = h, \text{ for all } h \in G \right\}, \tag{1.14}$$

and the *center* of a Lie algebra $\mathfrak{g}$ is defined as

$$Z(\mathfrak{g}) := \left\{ X \in \mathfrak{g} : [X, Y] = 0, \text{ for all } Y \in \mathfrak{g} \right\}. \tag{1.15}$$

The following result relates the centers of a Lie group and of its Lie algebra.

Corollary 1.57. *Let G be a connected Lie group. Then the following hold:*

(i) $Z(G) = \ker \mathrm{Ad}$;
(ii) $Z(G)$ *is a normal Lie subgroup of* G;
(iii) $Z(\mathfrak{g}) = \ker \mathrm{ad}$;
(iv) $Z(\mathfrak{g})$ *is the Lie algebra of* $Z(G)$.

Proof. First, we verify that $Z(G) = \ker \mathrm{Ad}$. If $g \in Z(G)$, clearly $\mathrm{Ad}(g) = \mathrm{id}$. Conversely, let $g \in \ker \mathrm{Ad}$. It follows from (1.10) that $g \exp(tX) g^{-1} = \exp(tX)$, for all $X \in \mathfrak{g}$. Hence g commutes with the elements of a neighborhood of the identity $e \in G$. From Proposition 1.20, we conclude that $g \in Z(G)$, proving (i).

Item (ii) follows from (i) and Lemma 1.48. Item (iii) follows directly from Proposition 1.37. Since $\mathrm{d}(\mathrm{Ad})_e = \mathrm{ad}$, (iv) follows directly from (iii) in Lemma 1.48. □

Remark 1.58. The above result gives an alternative proof of an assertion in Proposition 1.39, that G is abelian if $\mathfrak{g}$ is abelian. Indeed, if $[X,Y]=0$ for all $X,Y\in\mathfrak{g}$, then $Z(\mathfrak{g})=\mathfrak{g}$. Thus, from (iv) in Corollary 1.57, $Z(G)$ is open in G and it follows from Proposition 1.20 that $G=Z(G)$, hence G is abelian.

Exercise 1.59 ($\star$). In this exercise, we generalize the concept of *center* of a Lie group G and Lie algebra $\mathfrak{g}$, to the notion *centralizer* of subsets of G and $\mathfrak{g}$. Define the *centralizer* of a subset S in G as

$$Z_G(S) := \big\{g\in G : g\,h\,g^{-1}=h, \text{ for all } h\in S\big\},$$

and the *centralizer* of a subset $\mathfrak{s}$ in $\mathfrak{g}$ as

$$Z_{\mathfrak{g}}(\mathfrak{s}) := \big\{X\in\mathfrak{g} : [X,Y]=0, \text{ for all } Y\in\mathfrak{s}\big\}.$$

Note that $Z_G(G)=Z(G)$ and $Z_{\mathfrak{g}}(\mathfrak{g})=Z(\mathfrak{g})$.

Prove that $Z_G(S)$ is a Lie subgroup of G and $Z_{\mathfrak{g}}(\mathfrak{s})$ is a Lie subalgebra of $\mathfrak{g}$. Moreover, if G is connected, H is a Lie subgroup of G and $\mathfrak{h}$ is the corresponding Lie subalgebra of $\mathfrak{g}$, then the following hold (cf. Corollary 1.57):

(i) $Z_G(H)=\{g\in G : \mathrm{Ad}(g)X=X \text{ for all } X\in\mathfrak{h}\}$;
(ii) $Z_{\mathfrak{g}}(\mathfrak{h})=\{X\in\mathfrak{g} : [X,Y]=0 \text{ for all } Y\in\mathfrak{h}\}$;
(iii) $Z_{\mathfrak{g}}(\mathfrak{h})$ is the Lie algebra of $Z_G(H)$.

Exercise 1.60 ($\star$). In this exercise, we discuss relations between $\mathrm{U}(n)$ and $\mathrm{SU}(n)$.

(i) Show that the centers of $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are respectively given by

$$Z(\mathrm{U}(n))=\{z\,\mathrm{id} : z\in\mathbb{C}, |z|=1\}\cong S^1 \text{ and } Z(\mathrm{SU}(n))=\{z\,\mathrm{id} : z^n=1\}\cong\mathbb{Z}_n.$$

In particular, conclude that the Lie groups $\mathrm{U}(n)$ and $\mathrm{SU}(n)\times S^1$ are not isomorphic, since their centers are not isomorphic;
(ii) Show that the multiplication map $Z(\mathrm{U}(n))\times\mathrm{SU}(n)\ni(z\,\mathrm{id},A)\mapsto zA\in\mathrm{U}(n)$ is a Lie group homomorphism, and also an n-fold covering $S^1\times\mathrm{SU}(n)\to\mathrm{U}(n)$;
(iii) Find a Lie group homomorphism $\varphi\colon \mathrm{U}(n)\to S^1$ such that $\ker\varphi=\mathrm{SU}(n)$ and $\varphi\circ\iota=\mathrm{id}$, where $\iota\colon S^1\to\mathrm{U}(n)$ is the inclusion with image consisting of diagonal matrices with $e^{i\theta}$ in the upper left corner and 1 in the rest of the diagonal. Conclude that $\mathrm{U}(n)\cong\mathrm{SU}(n)\rtimes S^1$ is a *semidirect product* of $\mathrm{SU}(n)$ by S^1.

Exercise 1.61 ($\star$). Let G be a Lie group and H a closed Lie subgroup. Define the *normalizer* of H in G as

$$N(H) := \big\{g\in G : g\,H\,g^{-1}=H\big\}, \tag{1.16}$$

which is also sometimes denoted $N_G(H)$. Prove that $N(H)$ is a closed Lie subgroup of G. Assuming that H is connected and denoting its Lie algebra by $\mathfrak{h}$, prove that the following hold:

(i) $N(H) = \{g \in G : \mathrm{Ad}(g)X \in \mathfrak{h} \text{ for all } X \in \mathfrak{h}\}$;
(ii) The Lie algebra of $N(H)$ is $\mathfrak{n} = \{X \in \mathfrak{g} : [X,Y] \in \mathfrak{h} \text{ for all } Y \in \mathfrak{h}\}$.

Hint: Use (1.8), (1.10) and (1.12). For a solution, see the proof of Proposition 2.37.