Chapter 2
Lie Groups with Bi-invariant Metrics

This chapter deals with Lie groups with special types of Riemannian metrics: *bi-invariant* metrics. Every compact Lie group admits one such metric (see Proposition 2.24), which plays a very important role in the study of its geometry. In what follows, we use tools from Riemannian geometry to give concise proofs of several classical results on compact Lie groups.

We begin by reviewing some auxiliary facts of Riemannian geometry. Basic results on bi-invariant metrics and Killing forms are then discussed, e.g., we prove that the exponential map of a compact Lie group is surjective (Theorem 2.27), and we show that a semisimple Lie group is compact if and only if its Killing form is negative-definite (Theorem 2.35). We also prove that a simply-connected Lie group admits a bi-invariant metric if and only if it is a product of a compact Lie group with a vector space (Theorem 2.45). Finally, we prove that if the Lie algebra of a compact Lie group G is simple, then the bi-invariant metric on G is unique up to rescaling (Proposition 2.48).

Further suggested readings on the contents of this chapter are Fegan [85], Grove [106], Ise and Takeuchi [133], Milnor [158, 159], Onishchik [179] and Petersen [183], which inspired the present text.

2.1 Basic Facts of Riemannian Geometry

The main objective of this section is to review basic results of Riemannian geometry that are used in the following chapters. Proofs of most results in this section are omitted, and can be found in any standard textbook on Riemannian geometry, such as do Carmo [61], Petersen [183] or Lee [152].

A *Riemannian manifold* is a smooth manifold M endowed with a *(Riemannian) metric*, i.e., a $(0,2)$-tensor field g on M that is

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(i) Symmetric: $\mathrm{g}(X,Y) = \mathrm{g}(Y,X)$, for all $X,Y \in TM$;
(ii) Positive-definite: $\mathrm{g}(X,X) > 0$, if $X \neq 0$.

This means that a metric determines an inner product g_p on each tangent space T_pM. For this reason, we sometimes write $\langle X,Y\rangle = \mathrm{g}(X,Y)$ and $\|X\|^2 = \mathrm{g}(X,X)$, if the metric g is evident from the context. It is not difficult to prove that every manifold admits a Riemannian metric, using partitions of unity. As indicated below, Riemannian metrics provide a way to measure distances, angles, curvature, and other geometric properties.

It is possible to associate to each metric a so-called *connection*. Such map allows to *parallel translate* vectors along curves (see Proposition 2.7), *connecting* tangent spaces of M at different points. It is actually possible to define connections on *any* vector bundles over a manifold.

Definition 2.1. A *linear connection* on a Riemannian manifold (M,g) is a map

$$\nabla\colon \mathfrak{X}(M)\times\mathfrak{X}(M) \ni (X,Y) \longmapsto \nabla_X Y \in \mathfrak{X}(M),$$

satisfying the following properties:

(i) $\nabla_X Y$ is $C^\infty(M)$-linear in X, i.e., for all $f,g \in C^\infty(M)$,

$$\nabla_{fX_1+gX_2} Y = f\nabla_{X_1}Y + g\nabla_{X_2}Y;$$

(ii) $\nabla_X Y$ is $\mathbb{R}$-linear in Y, i.e., for all $a,b\in\mathbb{R}$,

$$\nabla_X(aY_1+bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2;$$

(iii) ∇ satisfies the Leibniz rule, i.e., for all $f \in C^\infty(M)$,

$$\nabla_X(fY) = f\nabla_X Y + (Xf)Y.$$

Moreover, a linear connection is said to be *compatible with the metric* g if

$$X\mathrm{g}(Y,Z) = \mathrm{g}(\nabla_X Y,Z) + \mathrm{g}(Y,\nabla_X Z), \quad X,Y,Z \in \mathfrak{X}(M).$$

It turns out that requiring a connection to be compatible with the metric does not determine a unique connection on (M,g). To this purpose, define the *torsion tensor* of ∇ to be the $(1,2)$-tensor field given by $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$. A connection ∇ is said to be *symmetric* if its torsion vanishes identically, that is, if $[X,Y] = \nabla_X Y - \nabla_Y X$ for all $X,Y \in \mathfrak{X}(M)$.

Exercise 2.2. Let M be an embedded surface in $\mathbb{R}^3$ with the induced metric, i.e., $\mathrm{g} = i^*\mathrm{g}_0$ where g_0 is the Euclidean metric and $i\colon M \to \mathbb{R}^3$ is the inclusion. Let $\overline{\nabla}$ be the Euclidean derivative. Set $\nabla_X Y := (\overline{\nabla}_{\overline{X}}\overline{Y})^\top$, i.e., the tangent part of $\overline{\nabla}_{\overline{X}}\overline{Y}$ where $\overline{X}$ and $\overline{Y}$ are local extension of the vector field X and Y. Show that ∇ is a symmetric connection compatible with the metric g.

Hint: One can prove that the connection is symmetric using Remark A.11.

Levi-Civita Theorem 2.3. *On a Riemannian manifold* (M, g)*, there exists a unique linear connection* ∇ *that is compatible with* g *and symmetric.*

The key fact on the proof of this theorem is the equation known as *connection formula*, or *Koszul formula*. It exhibits the natural candidate to the desired connection, and shows that it is uniquely determined by the metric:

$$\begin{aligned} \langle \nabla_Y X, Z\rangle = \tfrac{1}{2}\Big(& X\langle Y,Z\rangle - Z\langle X,Y\rangle + Y\langle Z,X\rangle \\ & - \langle [X,Y],Z\rangle - \langle [X,Z],Y\rangle - \langle [Y,Z],X\rangle \Big). \end{aligned} \tag{2.1}$$

The unique symmetric linear connection compatible with the metric is called the *Levi-Civita connection*, or *Riemannian connection*, and we refer to it simply as *connection*. Using this connection, one can differentiate vector fields on a Riemannian manifold (M, g) as we describe next.

Proposition 2.4. *Let* M *be a manifold with linear connection* ∇ *and* $\gamma\colon I \to M$ *be a smooth curve. Let* $\Gamma(\gamma^* TM)$ *denote the set of smooth vector fields along* γ*. There exists a unique correspondence that, to each* $X \in \Gamma(\gamma^* TM)$ *associates* $\frac{\mathrm{D}}{\mathrm{d}t} X \in \Gamma(\gamma^* TM)$*, called the* covariant derivative *of* X *along* γ*, satisfying:*

(i) *Linearity, i.e., for all* $X, Y \in \Gamma(\gamma^* TM)$*,*

$$\frac{\mathrm{D}}{\mathrm{d}t}(X+Y) = \frac{\mathrm{D}}{\mathrm{d}t}X + \frac{\mathrm{D}}{\mathrm{d}t}Y;$$

(ii) *Leibniz rule, i.e., for all* $X \in \Gamma(\gamma^* TM), f \in C^\infty(I)$*,*

$$\frac{\mathrm{D}}{\mathrm{d}t}(fX) = \frac{\mathrm{d}f}{\mathrm{d}t}X + f\frac{\mathrm{D}}{\mathrm{d}t}X;$$

(iii) *If* X *is induced from a vector field* $\widetilde{X} \in \mathfrak{X}(M)$*, that is* $X(t) = \widetilde{X}(\gamma(t))$*, then* $\frac{\mathrm{D}}{\mathrm{d}t}X = \nabla_{\gamma'}\widetilde{X}$*.*

Note that to each linear connection on M, the proposition above gives a covariant derivative operator for vector fields along γ. As mentioned before, we only consider the Levi-Civita connection, hence the covariant derivative is uniquely defined.

Equipped with this notion, it is possible to define the *acceleration* of a curve as the covariant derivative of its tangent vector field, and *geodesics* as curves with null acceleration. More precisely, $\gamma\colon I \to M$ is a *geodesic* if $\frac{\mathrm{D}}{\mathrm{d}t}\gamma' = 0$. It is often convenient to reparametrize a geodesic γ to become a *unit speed geodesic* $\widetilde{\gamma}$, that is, $\|\widetilde{\gamma}'\| = 1$ and $\widetilde{\gamma}$ has the same image as γ.

Writing a local expression for the covariant derivative, it is easy to see that a curve is a geodesic if and only if it satisfies a second-order system of ODEs, called the *geodesic equation*. Hence, applying the classical ODE result that guarantees existence and uniqueness of solutions, one can prove the following.

Theorem 2.5. *For any $p \in M$, $t_0 \in \mathbb{R}$ and $v \in T_pM$, there exist an open interval $I \subset \mathbb{R}$ containing t_0 and a geodesic $\gamma\colon I \to M$ satisfying the initial conditions $\gamma(t_0) = p$ and $\gamma'(t_0) = v$. In addition, any two geodesics with those initial conditions agree on their common domain.*

From uniqueness of the solution, it is possible to obtain the existence of a geodesic with *maximal domain* for a prescribed initial data. The restriction of a geodesic to a bounded subset of its maximal domain is called a *geodesic segment*.

Exercise 2.6. Let M be an embedded surface in $\mathbb{R}^3$, with the induced connection (defined in Exercise 2.2). Let V be a plane, and assume that, for all $p \in M \cap V$, the normal vector $\xi(p)$ is tangent to V. Show that the intersection $M \cap V$ can be locally parametrized by a geodesic segment on M. Conclude that the profile curve of a surface of revolution is a geodesic (up to reparametrization). In particular, conclude that great circles are the geodesics of the round sphere $S^2 \subset \mathbb{R}^3$.

Hint: Consider $\gamma \in V \cap M$ such that $\|\gamma'(t)\| = 1$. Note that $\langle \gamma''(t), \gamma'(t)\rangle = 0$, and hence $\gamma''(t)$ is normal to γ and contained in V. Since ξ is also normal to γ and contained in V, conclude that $\gamma'' = \lambda\xi$ and therefore $\frac{\mathrm{D}}{\mathrm{d}t}\gamma' = 0$.

Another construction that involves covariant differentiation along curves is parallel translation. A vector field $X \in \Gamma(\gamma^*TM)$ is said to be *parallel along* γ if $\frac{\mathrm{D}}{\mathrm{d}t}X = 0$. Thus, a geodesic γ can be characterized as a curve whose tangent field γ' is parallel along γ. A vector field is called *parallel* if it is parallel along every curve.

Proposition 2.7. *Let $\gamma\colon I \to M$ be a curve, $t_0 \in I$ and $v_0 \in T_{\gamma(t_0)}M$. There exists a unique parallel vector field X along γ such that $X(t_0) = v_0$.*

This vector field is called the *parallel translate* of v_0 along γ. Once more, the proof depends on basic ODE results.

An important question after having existence and uniqueness of geodesics is how geodesics change under perturbations of the initial data, which leads to the definition of the *geodesic flow* of a Riemannian manifold. This is the flow

$$\varphi^{\mathscr{G}}\colon U \subset \mathbb{R} \times TM \to TM$$

defined in an open subset U of $\mathbb{R} \times TM$ that contains $\{0\} \times TM$, of the unique vector field $\mathscr{G}$ on the tangent bundle, i.e., $\mathscr{G} \in \mathfrak{X}(TM)$, whose integral curves are of the form $t \mapsto (\gamma(t), \gamma'(t))$, where γ is a geodesic in M. This means that:

(i) $\gamma(t) = \pi \circ \varphi^{\mathscr{G}}(t, (p, v))$ is the geodesic with initial conditions $\gamma(0) = p$ and $\gamma'(0) = v$, where $\pi\colon TM \to M$ is the bundle projection;
(ii) $\varphi^{\mathscr{G}}(t, (p, cv)) = \varphi^{\mathscr{G}}(ct, (p, v))$, for all $c \in \mathbb{R}$ such that this equation makes sense.

In fact, supposing that such vector field $\mathscr{G}$ exists, one can obtain conditions in local coordinates that this field must satisfy (corresponding to the geodesic equation). Defining the vector field as its solutions, one may use Theorem A.14 to guarantee existence and smoothness of $\varphi^{\mathscr{G}}$.

We now proceed to define a similar concept to the Lie exponential map, which coincides with it when considering appropriate (bi-invariant) Riemannian metrics on compact Lie groups, see Theorem 2.27.

Definition 2.8. The *(Riemannian) exponential map* at $p \in M$ is the map

$$\exp_p \colon B_\varepsilon(0) \subset T_pM \to M, \quad \exp_p(v) := \pi \circ \varphi^{\mathscr{G}}(1,(p,v)).$$

Smoothness of $\exp_p$ follows immediately from Theorem A.14. Using the Inverse Function Theorem, one can verify that for any $p \in M$, there exist a neighborhood V of the origin in T_pM and a neighborhood U of $p \in M$, such that $\exp_p|_V \colon V \to U$ is a diffeomorphism. Such neighborhood U is called a *normal neighborhood* of p.

Define the *(Riemannian) distance* $\operatorname{dist}(p,q)$ for any pair of points $p,q \in M$ to be the infimum of lengths of all piecewise smooth curve segments joining p and q, where the *length* of a curve segment $\gamma \colon [a,b] \to M$ is defined as

$$\ell(\gamma) := \int_a^b \sqrt{\mathrm{g}(\gamma'(t),\gamma'(t))}\,\mathrm{d}t.$$

Then (M,dist) is a metric space, and the topology induced by this distance coincides with the original topology from the atlas of M. It is a very important fact that geodesics *locally minimize* ℓ among piecewise smooth curves. Geodesic segments $\gamma \colon [a,b] \to M$ that globally minimize distance, i.e., such that $\operatorname{dist}(\gamma(t_1),\gamma(t_2)) = |t_1 - t_2|$ for all $t_1,t_2 \in [a,b]$, are called *minimal*.

A Riemannian manifold is called *geodesically complete* if the maximal domain of definition of all geodesics is $\mathbb{R}$. It is not difficult to see that compactness is a sufficient condition for a manifold to be complete. The following important result states that all completeness notions for a Riemannian manifold are equivalent.

Hopf-Rinow Theorem 2.9. *Let (M,g) be a connected Riemannian manifold and $p \in M$. The following statements are equivalent:*

(i) $\exp_p$ *is globally defined, that is,* $\exp_p \colon T_pM \to M$*;*
(ii) *M is geodesically complete;*
(iii) (M,dist) *is a complete metric space;*
(iv) *Every closed bounded set in M is compact.*

If M satisfies any (hence all) of the above items, each two points of M can be joined by a minimal geodesic segment. In particular, for each $x \in M$ the exponential map $\exp_x \colon T_xM \to M$ *is surjective.*

Exercise 2.10. Give an example of a Riemannian manifold which is not complete, but such that any pair of points can be joined by a minimal geodesic segment.

Consider M and N manifolds and $f \colon M \to N$ a smooth map. Recall that any $(0,s)$-tensor τ on N may be *pulled back* by f, the result being a $(0,s)$-tensor $f^*\tau$ on M, see (A.6). A diffeomorphism $f \colon (M,\mathrm{g}^M) \to (N,\mathrm{g}^N)$ satisfying $f^*\mathrm{g}^N = \mathrm{g}^M$, that is, $g^M_p(X,Y) = g^N_{f(p)}(\mathrm{d}f_pX, \mathrm{d}f_pY)$ for all $p \in M$ and $X,Y \in T_pM$, is called a *(Riemannian) isometry*. It is easy to show that Riemannian isometries map geodesics to geodesics, and hence preserve distance, that is, are also metric isometries. Conversely, metric isometries of a smooth Riemannian manifold are Riemannian isometries. Thus, we simply refer to these maps as *isometries*.

From the above, it also follows that isometries *commute* with the exponential map, in the sense that if $f\colon M \to M$ is an isometry, then:

$$f(\exp_p(v)) = \exp_{f(p)}(\mathrm{d}f_p v) \tag{2.2}$$

for all $p \in M$ and $v \in T_pM$ such that $\exp_p(v)$ is defined. This allows to prove that isometries of a connected manifold are determined by its value and derivative at a single point.

Exercise 2.11. Suppose $f_1\colon M \to M$ and $f_2\colon M \to M$ are isometries of a connected Riemannian manifold (M,g), such that there exists $p \in M$ with $f_1(p) = f_2(p)$ and $\mathrm{d}(f_1)_p = \mathrm{d}(f_2)_p$. Prove that $f_1(x) = f_2(x)$ for all $x \in M$.

Hint: Consider the closed subset $S = \{x \in M : f_1(x) = f_2(x) \text{ and } \mathrm{d}(f_1)_x = \mathrm{d}(f_2)_x\}$. Use (2.2) to prove that S is also open, and hence $S = M$, since M is connected.

Isometries of a Riemannian manifold (M,g) clearly form a group, denoted $\mathrm{Iso}(M,g)$, where the operation is given by composition. A fundamental result, due to Myers and Steenrod [173], is that this is a Lie group that acts smoothly on M.

Myers-Steenrod Theorem 2.12. *Let (M,g) be a Riemannian manifold. Any closed subgroup of $\mathrm{Iso}(M,g)$ in the compact-open topology is a Lie group. In particular, $\mathrm{Iso}(M,g)$ is a Lie group.*

Remark 2.13. A subset $G \subset \mathrm{Iso}(M,g)$ is closed in the compact-open topology if given any sequence $\{f_n\}$ of isometries in G that converges uniformly in compact subsets to a continuous map $f\colon M \to M$, then $f \in G$.

We now mention a special class of vector fields on M, closely related to $\mathrm{Iso}(M,g)$. A *Killing vector field* is a vector field whose local flow is a local isometry. Alternatively, Killing vector fields are characterized by the following property.

Proposition 2.14. *A vector field $X \in \mathfrak{X}(M)$ is a Killing vector field if and only if*

$$g(\nabla_Y X, Z) = -g(\nabla_Z X, Y), \quad \text{for all } Y,Z \in \mathfrak{X}(M).$$

Theorem 2.15. *The set $\mathfrak{iso}(M,g)$ of Killing fields on (M,g) is a Lie algebra. In addition, if (M,g) is complete, then $\mathfrak{iso}(M,g)$ is the Lie algebra of $\mathrm{Iso}(M,g)$.*

An essential concept in Riemannian geometry is that of *curvature*. The *curvature tensor* on (M,g) is defined as the $(1,3)$-tensor field given by the following expression,[1] for all $X,Y,Z \in \mathfrak{X}(M)$.

[1] We remark that some texts use the opposite sign convention for R, as there is no standard choice.

$$R(X,Y)Z := \nabla_{[X,Y]}Z - \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z$$

It is possible to use the metric to deal with this tensor as a $(0,4)$-tensor, given by

$$R(X,Y,Z,W) := \mathrm{g}(R(X,Y)Z,W). \tag{2.3}$$

There are many important symmetries of this tensor that we state below. For each $X,Y,Z,W \in \mathfrak{X}(M)$,

(i) R is skew-symmetric in the first two and last two entries:

$$R(X,Y,Z,W) = -R(Y,X,Z,W) = R(Y,X,W,Z);$$

(ii) R is symmetric in the first and last pairs of entries:

$$R(X,Y,Z,W) = R(Z,W,X,Y);$$

(iii) R satisfies a cyclic permutation property, called the *first Bianchi identity*:

$$R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0.$$

Using the curvature tensor, we can define the *sectional curvature* of the plane spanned by (linearly independent) vectors X and Y as

$$\sec(X,Y) := \frac{R(X,Y,X,Y)}{\mathrm{g}(X,X)\mathrm{g}(Y,Y) - \mathrm{g}(X,Y)^2}. \tag{2.4}$$

It is possible to verify that $\sec(X,Y)$ depends only on the plane σ spanned by X and Y, and not specifically on the given basis $\{X,Y\}$ of σ. For this reason, we sometimes denote (2.4) by $\sec(\sigma)$, where $\sigma = \operatorname{span}\{X,Y\} \subset T_pM$.

There are several interpretations of curvature, the most naive being that it measures the failure of second covariant derivatives to commute. The curvature tensor is also part of the *Jacobi equation* along a geodesic γ, given by

$$\frac{\mathrm{D}}{\mathrm{d}t}\frac{\mathrm{D}}{\mathrm{d}t}J + R\big(\gamma'(t), J(t)\big)\gamma'(t) = 0. \tag{2.5}$$

This is an ODE whose solutions J are vector fields along γ, called *Jacobi fields*. A Jacobi field J along γ is the velocity (or variational field) of a variation of γ by geodesics. More precisely, $J(t) = \frac{\partial}{\partial s}\alpha(s,t)\big|_{s=0}$ where $\alpha\colon (-\varepsilon,\varepsilon)\times[a,b] \to M$ is a piecewise smooth map, such that $\alpha(0,t) = \gamma(t)$ and $\alpha(s,t)$ is a geodesic for each fixed $s \in (-\varepsilon,\varepsilon)$. Jacobi fields describe how quickly geodesics with the same starting point and different directions move away from each other.

Exercise 2.16 ($\star$). Let M be a surface with constant sectional curvature k, and let γ be a unit speed geodesic on M. Consider a variation $\alpha(s,t)$ by geodesics, such that

$\alpha(s,0)=\gamma(0)$ for all s. Consider the Jacobi field $J(t):=\frac{\partial}{\partial s}\alpha(s,t)\big|_{s=0}$, and assume that $\frac{\mathrm{D}}{\mathrm{d}t}J(0)=w$ is orthogonal to γ and $\|w\|=1$. Let $w(t)$ be the parallel translate of w along $\gamma(t)$. Prove that $J(t)=\mathrm{sn}_k(t)w(t)$, where sn_k is the function given by

$$\mathrm{sn}_k(t):=\begin{cases}\frac{\sin(t\sqrt{k})}{\sqrt{k}} & \text{if } k>0,\\ t & \text{if } k=0,\\ \frac{\sinh(t\sqrt{-k})}{\sqrt{-k}} & \text{if } k<0.\end{cases}$$

Hint: Write $J(t)=f_1(t)\gamma'(t)+f_2(t)w(t)$ and use the properties of the curvature tensor R described above, the hypothesis that $\langle\frac{\mathrm{D}}{\mathrm{d}t}J(0),\gamma'(0)\rangle=0$, the fact that $\langle w(t),\gamma'(t)\rangle=0$ (because parallel translation is an isometry), and unicity of solutions to ODEs, to conclude that $f_1\equiv 0$ and $f_2(t)=\mathrm{sn}_k(t)$.

Remark 2.17. The above statement about Jacobi fields also holds on more general *space forms*, i.e., Riemannian manifolds $M(k)$ that have constant sectional curvature k, since their curvature tensor satisfies $R(X,Y,Z,W)=k(\langle X,Z\rangle\langle Y,W\rangle-\langle Y,Z\rangle\langle X,W\rangle)$. If a space form $M(k)$ is simply-connected and has dimension n, then it is isometric to the sphere $S^n(1/\sqrt{k})$, the Euclidean space $\mathbb{R}^n$ or the hyperbolic space $H^n(1/\sqrt{-k})$, according to the cases $k>0$, $k=0$ and $k<0$ respectively.

As illustrated by the above exercise, on manifolds with nonnegative sectional curvature ($\sec\geq 0$), the distance between geodesics starting at the same point grows *slower* than in flat space; in other words, the exponential map $\exp_p$ is distance nonincreasing. Similarly, on manifolds with nonpositive sectional curvature ($\sec\leq 0$), distance between geodesics starting at the same point grows *faster* than in flat space; which means that $\exp_p$ is distance nondecreasing. Curvature also describes how parallel transport along short loops differs from the identity. More details on these interpretations can be found in Grove [106], Jost [135] and Petersen [183].

Besides analyzing sectional curvatures (which actually fully determine the curvature tensor R), it is convenient to summarize information contained in R by constructing simpler tensors. The *Ricci tensor* is a $(0,2)$-tensor field defined as a trace of the curvature tensor R. More precisely, if $\{e_1,\dots,e_n\}$ is an orthonormal basis of T_pM,

$$\mathrm{Ric}(X,Y):=\mathrm{tr}\big(R(X,\cdot)Y\big)=\sum_{i=1}^n R(X,e_i,Y,e_i).$$

The *scalar curvature* is a function $\mathrm{scal}\colon M\to\mathbb{R}$ given by the trace of Ric, i.e.,

$$\mathrm{scal}:=\mathrm{tr}\big(\mathrm{Ric}\big)=\sum_{j=1}^n \mathrm{Ric}(e_j,e_j)=2\sum_{1\leq i<j\leq n}R(e_i,e_j,e_i,e_j)=2\sum_{1\leq i<j\leq n}\sec(e_i,e_j).$$

The curvatures Ric and scal respectively encode information on the volume distortion and volume defect of small balls in (M,g), compared with a space form.

A metric is called *Einstein* if it is proportional to its Ricci tensor:

$$\mathrm{Ric}(X,Y) = \lambda\, \mathrm{g}(X,Y). \tag{2.6}$$

In this case, the number $\lambda \in \mathbb{R}$ is called the *Einstein constant* of g. It is easy to see that if (M,g) has constant sectional curvature $\sec = k$, then it is Einstein, with constant $\lambda = (n-1)k$, where $n = \dim M$. Moreover, any Einstein metric with constant λ has constant scalar curvature $\mathrm{scal} = n\lambda$.

Remark 2.18. Einstein metrics originate from General Relativity, as solutions to the Einstein equations in vacuum, $\mathrm{Ric} = \left(\frac{1}{2}\,\mathrm{scal} - \Lambda\right)\mathrm{g}$, where Λ is the cosmological constant. Details on this equation and its impact in Physics can be found in [33, 160]. As a curiosity, the reason why Riemannian metrics are denoted g also comes from General Relativity, since, in this context, g is interpreted as a gravitational field.

We now state a classical result that establishes a link between the curvature and topology of a manifold. Recall that the diameter $\mathrm{diam}(X)$ of a metric space X is the smallest number d such that any two points in X are always at distance $\leq d$.

Bonnet-Myers Theorem 2.19. *Let (M,g) be a connected complete n-dimensional Riemannian manifold, with $n \geq 2$. Assume* $\mathrm{Ric} \geq (n-1)k\,\mathrm{g}$ *for some $k > 0$. Then:*

(i) $\mathrm{diam}(M) \leq \frac{\pi}{\sqrt{k}}$*; in particular, M is compact;*
(ii) *The universal covering of M is compact, hence $\pi_1(M)$ is finite.*

We end this section with a quick discussion of Riemannian *immersions* and *submersions*, which are dual types of maps between Riemannian manifolds (of potentially different dimensions), that generalize the notion of isometries.

Let (M,g) and $(\overline{M},\overline{\mathrm{g}})$ be Riemannian manifolds and $i\colon M \to \overline{M}$ be an immersion (recall Definition A.4). Then i is a *Riemannian immersion* if for all $p \in M$, $\mathrm{d}i_p$ is a linear isometry from T_pM onto its image, which is a subspace of $T_{i(p)}\overline{M}$. In other words, $\mathrm{g} = i^*\overline{\mathrm{g}}$. A vector field X on M can always be locally extended to a vector field $\overline{X}$ on $\overline{M}$. Let $\overline{\nabla}$ be the Levi-Civita connection of $(\overline{M},\overline{\mathrm{g}})$. It is easy to prove that the component of $\overline{\nabla}_{\overline{X}}\overline{Y}$ that is tangent to M, denoted $(\overline{\nabla}_{\overline{X}}\overline{Y})^\top$, gives the value of the Levi-Civita connection of (M,g) applied to vector fields X and Y along M, independent of the choice of local extensions $\overline{X}$ and $\overline{Y}$ (cf. Exercise 2.2). The difference between these connections is a bilinear symmetric map with values on the normal space to M, called the *second fundamental form* II of M:

$$\mathrm{II}_p\colon T_pM \times T_pM \to \nu_pM, \quad \mathrm{II}(X,Y)_p := (\overline{\nabla}_{\overline{X}}\overline{Y})_p - (\overline{\nabla}_{\overline{X}}\overline{Y})_p^\top,$$

where ν_pM denotes the *normal space* to M at p, i.e., such that $T_{i(p)}\overline{M} = T_pM \oplus \nu_pM$ is a $\overline{\mathrm{g}}$-orthogonal direct sum. For each normal vector ξ to M at p, we can also define a bilinear symmetric form II_ξ by taking the inner product of ξ and II,

$$(\mathrm{II}_\xi)_p\colon T_pM \times T_pM \to \mathbb{R}, \quad \mathrm{II}_\xi(X,Y)_p := \overline{\mathrm{g}}_p(\xi, \mathrm{II}(X,Y)).$$

If $\mathrm{I\!I}$ vanishes identically, then M is called a *totally geodesic* submanifold. This is equivalent to each geodesic of M being a geodesic of $\overline{M}$, since the connections ∇ and $\overline{\nabla}$ agree and hence so do the corresponding geodesic equations. Being totally geodesic is a very strong property; however, this class of submanifolds appears many times along this text (see, e.g., Exercise 2.29 and Proposition 3.93).

The relation between curvatures on M and $\overline{M}$ is given by the *Gauss equation*

$$R(X,Y,X,Y) = \overline{R}(\overline{X},\overline{Y},\overline{X},\overline{Y}) + \overline{\mathrm{g}}(\mathrm{I\!I}(X,X),\mathrm{I\!I}(Y,Y)) - \overline{\mathrm{g}}(\mathrm{I\!I}(X,Y),\mathrm{I\!I}(X,Y)), \quad (2.7)$$

where R and $\overline{R}$ denote the curvature tensors of M and $\overline{M}$, respectively. Assuming X,Y and $\overline{X},\overline{Y}$ are pairs of orthonormal vectors, the quantities $\sec(X,Y) = R(X,Y,X,Y)$ and $\overline{\sec}(\overline{X},\overline{Y}) = \overline{R}(\overline{X},\overline{Y},\overline{X},\overline{Y})$ are called *intrinsic* and *extrinsic* curvatures of the plane tangent to M spanned by these vectors. Note that both agree if M is totally geodesic.

Since $\mathrm{I\!I}_\xi$ is symmetric, there exists a self-adjoint operator $\mathscr{S}_\xi$ with respect to g, called the *shape operator*, satisfying

$$\mathrm{g}(\mathscr{S}_\xi X, Y) = \mathrm{I\!I}_\xi(X,Y). \quad (2.8)$$

It is not difficult to prove that $\mathscr{S}_\xi(X) = (-\overline{\nabla}_X \overline{\xi})^\top$ where $\overline{\xi}$ is any normal field that extends ξ. Eigenvalues and eigenvectors of $\mathscr{S}_\xi(X)$ are respectively called *principal curvatures* and *principal directions*; see Remark 2.21 for a geometric interpretation.

Exercise 2.20. Compute the principal curvatures and directions of a round sphere and a round cylinder embedded in $\mathbb{R}^3$.

Hint: Use the explicit description of the normal vectors fields (e.g., the normal to the unit sphere is $\xi(x,y,z) = (x,y,z)$) and apply the chain rule, i.e., $\mathscr{S}_\xi(X) = (-\overline{\nabla}_X \overline{\xi})^\top = -\overline{\nabla}_X \overline{\xi} = \frac{\mathrm{d}}{\mathrm{d}t}(\xi \circ \alpha)$, where α is a curve in the surface.

The second fundamental form also allows us to define the *mean curvature vector* $\mathbf{H}$ of the submanifold M, which is a section of $\nu(M)$ given by the trace of the second fundamental form, i.e.,

$$\mathbf{H}(p) := \sum_i \mathrm{I\!I}(e_i, e_i) \quad (2.9)$$

where $\{e_i\}$ is an orthonormal basis of T_pM. The length $\|\mathbf{H}\|$ of the mean curvature vector is called the *mean curvature* of M. A submanifold whose mean curvature is identically zero is called *minimal*. From the definitions, it is clear that totally geodesic submanifolds are automatically minimal, but the converse need not be true.

Remark 2.21. Let M be an embedded surface in $\overline{M} = \mathbb{R}^3$ with the induced metric. According to the Gauss equation (2.7), the sectional curvature of M coincides with *Gaussian curvature* of M, i.e., the product of eigenvalues $\lambda_1\lambda_2$ (principal curvatures) of the shape operator $\mathscr{S}_\xi(\cdot) = -\overline{\nabla}_{(\cdot)}\xi$, where ξ is a unitary normal vector to M. Notice that this is precisely the right-hand side of (2.7), since the determinant of $\mathrm{I\!I}$

is equal to the product of its eigenvalues. An important fact in differential geometry is that each embedded surface with nonzero Gaussian curvature is, up to rigid motions, locally given by the graph of $f(x_1,x_2) = \frac{1}{2}\left(\lambda_1 x_1^2 + \lambda_2 x_2^2\right) + O(\|x\|^3)$. In other words, if $\sec > 0$ (respectively $\sec < 0$) M is locally a small perturbation of elliptic (respectively hyperbolic) paraboloid.

The dual notion to an immersion is that of a submersion. Let $(\overline{M},\overline{\mathrm{g}})$ and (M,g) be Riemannian manifolds and $\pi\colon \overline{M} \to M$ be a submersion (recall Definition A.4). Then π is a *Riemannian submersion* if for all $p \in \overline{M}$, $\mathrm{d}\pi_p$ is a linear isometry from $(\ker \mathrm{d}\pi_p)^\perp$ onto $T_{\pi(p)}M$, where $(\ker \mathrm{d}\pi_p)^\perp$ is the subspace of $T_p\overline{M}$ given by the $\overline{\mathrm{g}}$-orthogonal complement to $\ker \mathrm{d}\pi_p$. The subspace

$$\mathscr{V}_p := \ker \mathrm{d}\pi_p \tag{2.10}$$

is called the *vertical space* at $p \in \overline{M}$, while

$$\mathscr{H}_p := (\ker \mathrm{d}\pi_p)^\perp = \left\{X \in T_p\overline{M} : \overline{\mathrm{g}}(X,Y) = 0 \text{ for all } Y \in \mathscr{V}_p\right\} \tag{2.11}$$

is called the *horizontal space* at $p \in \overline{M}$. The collections of vertical and horizontal spaces on $\overline{M}$ form two complementary distributions $\mathscr{V}$ and $\mathscr{H}$ on $\overline{M}$, that we accordingly call *vertical* and *horizontal distributions*. The vertical distribution is clearly integrable, since it is tangent to the foliation $\mathscr{F}_\pi = \{\pi^{-1}(x) : x \in M\}$, but the horizontal distribution may be nonintegrable. Again, denote by $\overline{\nabla}$ and ∇ the Levi-Civita connections on $\overline{M}$ and M, respectively. The two fundamental tensors T and A of the Riemannian submersion $\pi\colon \overline{M} \to M$ are $(1,2)$-tensors on $\overline{M}$, given by

$$\begin{aligned} T_X Y &= \left(\overline{\nabla}_{X^{\mathscr{V}}} Y^{\mathscr{V}}\right)^{\mathscr{H}} + \left(\overline{\nabla}_{X^{\mathscr{V}}} Y^{\mathscr{H}}\right)^{\mathscr{V}}, \\ A_X Y &= \left(\overline{\nabla}_{X^{\mathscr{H}}} Y^{\mathscr{H}}\right)^{\mathscr{V}} + \left(\overline{\nabla}_{X^{\mathscr{H}}} Y^{\mathscr{V}}\right)^{\mathscr{H}}, \end{aligned} \tag{2.12}$$

where $X^{\mathscr{V}}$ and $X^{\mathscr{H}}$ respectively denote vertical and horizontal components of a vector field X on $\overline{M}$. The restrictions of the tensors T and A to $\mathscr{V}$ and $\mathscr{H}$ can be interpreted respectively as the second fundamental form of the leaves of $\mathscr{F}_\pi$ and the integrability of the horizontal distribution $\mathscr{H}$. In particular, we say that a Riemannian submersion is *integrable* if the restriction of A to $\mathscr{H}$ vanishes identically.

Analogously to the local extension of vector fields for Riemannian immersions, there exists a horizontal lift property for Riemannian submersions. Namely, a vector field X on M can always be lifted to a horizontal vector field $\overline{X}$ on $\overline{M}$ that is π-related to X, i.e., such that $\mathrm{d}\pi(\overline{X}) = X \circ \pi$. Using this, it is not hard to see that if X and Y are vector fields on M, then

$$\overline{\nabla}_{\overline{X}}\overline{Y} - \overline{\nabla_X Y} = A_{\overline{X}}\overline{Y} = \tfrac{1}{2}[\overline{X},\overline{Y}]^{\mathscr{V}}, \tag{2.13}$$

see Petersen [183, p. 82]. The relation between curvatures on $\overline{M}$ and M was first studied by Gray [104] and O'Neill [177], who proved that

$$\begin{aligned} R(X,Y,X,Y) &= \overline{R}(\overline{X},\overline{Y},\overline{X},\overline{Y}) + 3\|A_{\overline{X}}\overline{Y}\|^2 \\ &= \overline{R}(\overline{X},\overline{Y},\overline{X},\overline{Y}) + \tfrac{3}{4}\|[\overline{X},\overline{Y}]^{\mathscr{V}}\|^2, \end{aligned} \tag{2.14}$$

for all vectors X and Y on M. The above is called the *Gray-O'Neill formula*. In particular, it follows that if $\overline{M}$ satisfies a certain lower sectional curvature bound, then so does M. Formulas for the other sectional curvatures of $\overline{M}$ (containing vertical directions), as well as the implications for Ricci and scalar curvatures, can be found in Besse [33, Chap 9].

2.2 Bi-invariant Metrics

The main goal of this section is to study the special Riemannian structure on Lie groups given by *bi-invariant* metrics. We mostly use $\langle\cdot,\cdot\rangle$ to denote metrics on Lie groups, to avoid any confusion between a metric $\mathrm{g}(\cdot,\cdot)$ and an element $g \in G$. Bi-invariant metrics on Lie groups are denoted $Q(\cdot,\cdot)$.

Definition 2.22. A Riemannian metric $\langle\cdot,\cdot\rangle$ on a Lie group G is *left-invariant* if L_g is an isometry for all $g \in G$, that is, if for all $g,h \in G$ and $X,Y \in T_hG$,

$$\langle \mathrm{d}(L_g)_h X, \mathrm{d}(L_g)_h Y\rangle_{gh} = \langle X,Y\rangle_h.$$

Similarly, *right-invariant* metrics are those for which the right translations R_g are isometries. Note that, given an inner product $\langle\cdot,\cdot\rangle_e$ in T_eG, it is possible to define a left-invariant metric on G by setting, for all $g \in G$ and $X,Y \in T_gG$,

$$\langle X,Y\rangle_g := \langle \mathrm{d}(L_{g^{-1}})_g X, \mathrm{d}(L_{g^{-1}})_g Y\rangle_e,$$

and the right-invariant case is analogous.

Definition 2.23. A *bi-invariant metric* Q on a Lie group G is a Riemannian metric that is simultaneously left- and right-invariant.

The natural extension of these concepts to k-forms is that a k-form $\omega \in \Omega^k(G)$ is *left-invariant* if it coincides with its pull-back by left translations, i.e., $L_g^*\omega = \omega$ for all $g \in G$. *Right-invariant* and *bi-invariant* forms are analogously defined. Once more, given any $\omega_e \in \Lambda^k(T_eG)$, it is possible to define a left-invariant k-form $\omega \in \Omega(G)$ by setting, for all $g \in G$ and $X_i \in T_gG$,

$$\omega_g(X_1,\dots,X_k) := \omega_e(\mathrm{d}(L_{g^{-1}})_g X_1,\dots,\mathrm{d}(L_{g^{-1}})_g X_k).$$

The right-invariant case is analogous.

Proposition 2.24. *Every compact Lie group admits a bi-invariant metric Q.*

Proof. Let ω be a right-invariant volume form[2] on G and $\langle\cdot,\cdot\rangle$ a right-invariant metric. Define for all $X,Y \in T_xG$,

$$Q(X,Y)_x := \int_G \langle \mathrm{d}L_g X, \mathrm{d}L_g Y\rangle_{gx}\,\omega.$$

First, we claim that Q is left-invariant. Fix $X,Y \in T_xG$ and consider the function $f\colon G \to \mathbb{R}$ given by $f(g) := \langle \mathrm{d}L_g X, \mathrm{d}L_g Y\rangle_{gx}$. Then,

$$\begin{aligned} Q(\mathrm{d}L_h X, \mathrm{d}L_h Y)_{hx} &= \int_G \langle \mathrm{d}L_g(\mathrm{d}L_h X), \mathrm{d}L_g(\mathrm{d}L_h Y)\rangle_{g(hx)}\,\omega \\ &= \int_G \langle \mathrm{d}L_{gh} X, \mathrm{d}L_{gh} Y\rangle_{(gh)x}\,\omega = \int_G f(gh)\,\omega \\ &= \int_G R_h^*(f\omega) = \int_G f\omega = \int_G \langle \mathrm{d}L_g X, \mathrm{d}L_g Y\rangle_{gx}\,\omega = Q(X,Y)_x, \end{aligned}$$

which proves that Q is left-invariant. We also have

$$\begin{aligned} Q(\mathrm{d}R_h X, \mathrm{d}R_h Y)_{xh} &= \int_G \langle \mathrm{d}L_g(\mathrm{d}R_h X), \mathrm{d}L_g(\mathrm{d}R_h Y)\rangle_{g(xh)}\,\omega \\ &= \int_G \langle \mathrm{d}R_h \mathrm{d}L_g X, \mathrm{d}R_h \mathrm{d}L_g Y\rangle_{(gx)h}\,\omega = \int_G \langle \mathrm{d}L_g X, \mathrm{d}L_g Y\rangle_{gx}\,\omega = Q(X,Y)_x, \end{aligned}$$

which proves that Q is right-invariant, concluding the proof. □

Exercise 2.25. Consider $\mathfrak{su}(n) = \{A \in \mathfrak{gl}(n,\mathbb{C}) : A^* + A = 0, \operatorname{tr} A = 0\}$, the Lie algebra of $\mathrm{SU}(n)$ (see Exercise 1.51). Verify that the inner product in $T_e\mathrm{SU}(n)$ defined by $Q(X,Y) = \frac{1}{2}\operatorname{Re}\operatorname{tr}(XY^*)$ can be extended to a bi-invariant metric.

Proposition 2.26. *Let G be a Lie group endowed with a bi-invariant metric Q, and $X,Y,Z \in \mathfrak{g}$. Then the following hold:*

(i) $Q([X,Y],Z) = -Q(Y,[X,Z])$;
(ii) $\nabla_X Y = \frac{1}{2}[X,Y]$;
(iii) $R(X,Y)Z = \frac{1}{4}[[X,Y],Z]$;
(iv) $R(X,Y,X,Y) = \frac{1}{4}\|[X,Y]\|^2$.

In particular, (G,Q) has nonnegative sectional curvature $\sec \geq 0$.

[2] i.e., $\omega \in \Omega^n(G)$ is a nonvanishing n-form, where $n = \dim G$, see Definition A.26.

Proof. Differentiating the formula $Q(\mathrm{Ad}(\exp(tX))Y, \mathrm{Ad}(\exp(tX))Z) = Q(Y,Z)$ it follows from Proposition 1.37 and (1.11) that

$$Q([X,Y],Z) + Q(Y,[X,Z]) = 0,$$

which proves (i). Furthermore, (ii) follows from the Koszul formula (2.1) using (i) and the fact that ∇ is symmetric.

To prove (iii), we use (ii) to compute $R(X,Y)Z$ as follows:

$$\begin{aligned}
R(X,Y)Z &= \nabla_{[X,Y]}Z - \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z \\
&= \tfrac{1}{2}[[X,Y],Z] - \tfrac{1}{2}\nabla_X [Y,Z] + \tfrac{1}{2}\nabla_Y [X,Z] \\
&= \tfrac{1}{2}[[X,Y],Z] - \tfrac{1}{4}[X,[Y,Z]] + \tfrac{1}{4}[Y,[X,Z]] \\
&= \tfrac{1}{4}[[X,Y],Z] + \tfrac{1}{4}\big([[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X]\big) \\
&= \tfrac{1}{4}[[X,Y],Z].
\end{aligned}$$

Finally, to prove (iv), we use (i) to verify that

$$\begin{aligned}
Q(R(X,Y)X,Y) &= \tfrac{1}{4}Q([[X,Y],X],Y) \\
&= -\tfrac{1}{4}Q([X,[X,Y]],Y) \\
&= \tfrac{1}{4}Q([X,Y],[X,Y]) \\
&= \tfrac{1}{4}\|[X,Y]\|^2.
\end{aligned}$$

□

Theorem 2.27. *The Lie exponential map and the Riemannian exponential map at the identity agree on Lie groups endowed with bi-invariant metrics. In particular, the Lie exponential map of a connected compact Lie group is surjective.*

Proof. Let G be a Lie group endowed with a bi-invariant metric and $X \in \mathfrak{g}$. To prove that the exponential maps coincide, it suffices to prove that the 1-parameter subgroup $\gamma\colon \mathbb{R} \to G$ given by $\gamma(t) = \exp(tX)$ is the geodesic with $\gamma(0) = e$ and $\gamma'(0) = X$. First, recall that γ is the integral curve of the left-invariant vector field X passing through $e \in G$ at $t = 0$, that is, $\gamma'(t) = X(\gamma(t))$ and $\gamma(0) = e$. Furthermore, from Proposition 2.26,

$$\frac{\mathrm{D}}{\mathrm{d}t}\gamma' = \frac{\mathrm{D}}{\mathrm{d}t}X(\gamma(t)) = \nabla_{\gamma'}X = \nabla_X X = \tfrac{1}{2}[X,X] = 0.$$

Therefore, γ is a geodesic and the Lie exponential map coincides with the Riemannian exponential map.

From Proposition 2.24, if G is compact, it admits a bi-invariant metric Q. Using that exponential maps coincide, and that the Lie exponential map is defined for all $X \in \mathfrak{g}$, it follows from the Hopf-Rinow Theorem 2.9 that (G,Q) is a complete Riemannian manifold. Thus, $\exp = \exp_e : T_eG \to G$ is surjective. □

Exercise 2.28. Use the fact that exp is not surjective in $\mathrm{SL}(2,\mathbb{R})$ to prove that $\mathrm{SL}(2,\mathbb{R})$ does not admit a metric such that the Lie exponential map and the Riemannian exponential map coincide in e.

Exercise 2.29. Let G be a compact Lie group endowed with a bi-invariant metric. Prove that each closed subgroup H is a totally geodesic submanifold.

In the next result, we prove that each Lie group G with bi-invariant metric is a *symmetric space*, i.e., for each $a \in G$ there exists an isometry I^a that reverses geodesics through a (see also Exercise 6.32).

Theorem 2.30. *Let G be a connected Lie group endowed with a bi-invariant metric. For each $g \in G$, set*

$$I^g : G \to G, \quad I^g(x) := gx^{-1}g.$$

Then I^g is an isometry that fixes g and reverses geodesics through g. In other words, $I^g \in \mathrm{Iso}(G)$ and if γ is a geodesic with $\gamma(0) = g$, then $I^g(\gamma(t)) = \gamma(-t)$.

Proof. Since $I^e(g) = g^{-1}$, the map $\mathrm{d}(I^e)_e : T_eG \to T_eG$ is the multiplication by -1, i.e., $\mathrm{d}(I^e)_e = -\mathrm{id}$. Hence it is an isometry of T_eG. Since $\mathrm{d}(I^e)_g = \mathrm{d}(R_{g^{-1}})_e \circ \mathrm{d}(I^e)_e \circ \mathrm{d}(L_{g^{-1}})_g$, for any $g \in G$, the map $\mathrm{d}(I^e)_g : T_gG \to T_{g^{-1}}G$ is also an isometry. Hence I^e is an isometry. It clearly reverses geodesics through e, and since $I^g = R_g I^e R_g^{-1}$, it follows that I^g is also an isometry that reverses geodesics through g. □

2.3 Killing Form and Semisimple Lie Algebras

To continue our study of bi-invariant metrics, we introduce the Killing form, which allows to establish a classical algebraic condition for compactness of Lie groups.

Definition 2.31. Let G be a Lie group and $X,Y \in \mathfrak{g}$. The *Killing form* of $\mathfrak{g}$ (also said Killing form of G) is the symmetric bilinear form

$$B(X,Y) := \mathrm{tr}\big(\mathrm{ad}(X)\mathrm{ad}(Y)\big).$$

If B is nondegenerate, then $\mathfrak{g}$ is said to be *semisimple*.

We provide in Theorem 2.41 other equivalent definitions of semisimplicity for a Lie algebra. A Lie group G is said to be *semisimple* if its Lie algebra $\mathfrak{g}$ is semisimple.

Proposition 2.32. *The Killing form is* Ad*-invariant, that is, for all $X,Y \in \mathfrak{g}$ and $g \in G$, it holds that $B(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) = B(X,Y)$.*

Proof. If $\varphi : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra automorphism, then $\mathrm{ad}(\varphi(X))\varphi(Y) = \varphi \circ \mathrm{ad}(X)Y$. Thus, $\mathrm{ad}(\varphi(X)) = \varphi \circ \mathrm{ad}(X) \circ \varphi^{-1}$ and hence

$$\begin{aligned} B(\varphi(X),\varphi(Y)) &= \operatorname{tr}\left(\operatorname{ad}(\varphi(X))\operatorname{ad}(\varphi(Y))\right) \\ &= \operatorname{tr}\left(\varphi\circ\operatorname{ad}(X)\operatorname{ad}(Y)\circ\varphi^{-1}\right) \\ &= \operatorname{tr}\left(\operatorname{ad}(X)\operatorname{ad}(Y)\right) \\ &= B(X,Y). \end{aligned}$$

Since $\mathrm{Ad}(g)$ is a Lie algebra automorphism, the proof is complete. □

Corollary 2.33. *Let G be a semisimple Lie group with negative-definite Killing form B. Then $-B$ is a bi-invariant metric.*

Remark 2.34. Let G be a Lie group endowed with a bi-invariant metric Q. From Proposition 2.26, it follows that

$$\operatorname{Ric}(X,Y) = \operatorname{tr} R(X,\cdot)Y = \operatorname{tr} \tfrac{1}{4}[[X,\cdot],Y] = -\tfrac{1}{4}\operatorname{tr}[Y,[X,\cdot]] = -\tfrac{1}{4}B(X,Y). \tag{2.15}$$

Therefore, the Ricci tensor of (G,Q) is independent of the bi-invariant metric Q.

Theorem 2.35. *Let G be a n-dimensional semisimple connected Lie group. Then G is compact if and only if its Killing form B is negative-definite.*

Proof. First, suppose that B is negative-definite. From Corollary 2.33, $-B$ is a bi-invariant metric on G. Hence, the Hopf-Rinow Theorem 2.9 and Theorem 2.30 imply that $(G,-B)$ is a complete Riemannian manifold, whose Ricci curvature satisfies (2.15). It follows from the Bonnet-Myers Theorem 2.19 that G is compact.

Conversely, suppose G is compact. From Proposition 2.24, it admits a bi-invariant metric Q. Hence, using item (i) of Propositions 2.26 and 1.37, it follows that, if $(e_1,\dots,e_n)$ is an orthonormal basis of $\mathfrak{g}$, then

$$\begin{aligned} B(X,X) &= \operatorname{tr}\left(\operatorname{ad}(X)\operatorname{ad}(X)\right) \\ &= \sum_{i=1}^{n} Q(\operatorname{ad}(X)\operatorname{ad}(X)e_i,e_i) \\ &= -\sum_{i=1}^{n} Q(\operatorname{ad}(X)e_i,\operatorname{ad}(X)e_i) \\ &= -\sum_{i=1}^{n} \|\operatorname{ad}(X)e_i\|^2 \leq 0. \end{aligned}$$

Note that, if there exists $X \neq 0$ such that $\|\operatorname{ad}(X)e_i\|^2 = 0$ for all i, then by definition of the Killing form, $B(Y,X) = 0$ for each Y. This would imply that B is degenerate, contradicting the fact that $\mathfrak{g}$ is semisimple. Thus, for each $X \neq 0$, we have $B(X,X) < 0$. Therefore, B is negative-definite. □

The next result follows directly from Corollary 2.33, Remark 2.34 and Theorem 2.35.

Corollary 2.36. *Let G be a semisimple compact connected Lie group with Killing form B. Then $(G,-B)$ is an Einstein manifold with positive Einstein constant $\lambda = \frac{1}{4}$.*

We conclude this section with a discussion on equivalent definitions of semisimple Lie algebras. The key concept in this discussion is that of ideal of a Lie algebra. A Lie subalgebra $\mathfrak{h}$ is an *ideal* of a Lie algebra $\mathfrak{g}$ if $[X,Y] \in \mathfrak{h}$, for all $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$. Using tools from the previous chapter, we now prove the following important relation between ideals of a Lie algebra and normal subgroups of a Lie group.

Proposition 2.37. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. If $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then the connected Lie subgroup H with Lie algebra $\mathfrak{h}$ is a normal subgroup of G. Conversely, if H is a normal Lie subgroup of G, then its Lie algebra $\mathfrak{h}$ is an ideal of* $\mathfrak{g}$.

Proof. Suppose $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, and let H be the connected Lie subgroup with Lie algebra $\mathfrak{h}$ given by Proposition 1.21. Then, by Proposition 1.37, we have that $\operatorname{ad}(X)Y \in \mathfrak{h}$ for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$. From (1.12), it follows that

$$\operatorname{Ad}(\exp(X))Y = \exp(\operatorname{ad}(X))Y = \sum_{k=0}^{\infty} \frac{\operatorname{ad}(X)^k}{k!} Y \in \mathfrak{h}.$$

On the other hand, from (1.10),

$$\exp(\operatorname{Ad}(\exp(X))Y) = \exp(X)\exp(Y)\exp(X)^{-1},$$

and hence $\exp(X)\exp(Y)\exp(X)^{-1} \in H$. From Proposition 1.20, it follows that H is a normal subgroup of G.

Conversely, suppose H is a normal subgroup of G, that is $gHg^{-1} = H$ for all $g \in G$. Then $\exp(t\operatorname{Ad}(g)X) = g\exp(tX)g^{-1} \in H$ for all $g \in G$ and $X \in \mathfrak{h}$. Differentiating at $t = 0$, we have that $\operatorname{Ad}(g)$ maps $\mathfrak{h}$ to itself. In particular, $\operatorname{Ad}(\exp(tX))Y \in \mathfrak{h}$ for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$. Differentiating again at $t = 0$ and using Proposition 1.37, we have that $[X,Y] \in \mathfrak{h}$ for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$, that is, $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. □

If an ideal $\mathfrak{h}$ has no ideals other than the trivial, $\{0\}$ and $\mathfrak{h}$, it is called *simple*. Following the usual convention, by *simple Lie algebras* we mean Lie subalgebras that are noncommutative simple ideals. We stress that simple ideals (which may be commutative) are not referred to as Lie algebras, but simply called *simple ideals*.

Given a Lie algebra $\mathfrak{g}$, consider the decreasing sequence of ideals

$$\mathfrak{g}^{(1)} := [\mathfrak{g},\mathfrak{g}], \quad \mathfrak{g}^{(2)} := [\mathfrak{g}^{(1)},\mathfrak{g}^{(1)}], \quad \dots \quad \mathfrak{g}^{(k)} := [\mathfrak{g}^{(k-1)},\mathfrak{g}^{(k-1)}], \quad \dots$$

If there exists a positive integer m such that $\mathfrak{g}^{(m)} = \{0\}$, then $\mathfrak{g}$ is said to be *solvable*.

Example 2.38. Consider the ideal of all $n \times n$ matrices (a_{ij}) over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, with $a_{ij} = 0$ if $i > j$. One can easily verify that this is a solvable Lie algebra. Other trivial examples are nilpotent Lie algebras.

It is possible to prove that every Lie algebra admits a maximal solvable ideal τ, called its *radical*. We recall some results whose proof can be found in Ise and Takeuchi [133].

Proposition 2.39. *The following hold:*

(i) *If $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then the Killing form $B_{\mathfrak{h}}$ of $\mathfrak{h}$ satisfies $B(X,Y) = B_{\mathfrak{h}}(X,Y)$, for all $X,Y \in \mathfrak{h}$;*
(ii) *If $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is direct sum of ideals, then $\mathfrak{g}_1$ is orthogonal to $\mathfrak{g}_2$ with respect to B. Thus B is the sum of the Killing forms B_1 and B_2 of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively.*

Cartan Theorem 2.40. *A Lie algebra $\mathfrak{g}$ is solvable if and only if $B(\mathfrak{g}, \mathfrak{g}^{(1)}) = \{0\}$. In particular, if B vanishes identically, then $\mathfrak{g}$ is solvable.*

We are now ready to present a theorem that gives equivalent definitions of semisimple Lie algebras.

Theorem 2.41. *Let $\mathfrak{g}$ be a Lie algebra with Killing form B. Then the following are equivalent:*

(i) *$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ is the direct sum of simple Lie algebras $\mathfrak{g}_i$ (i.e., noncommutative simple ideals);*
(ii) *$\mathfrak{g}$ has trivial radical $\tau = \{0\}$;*
(iii) *$\mathfrak{g}$ has no commutative ideal other than $\{0\}$;*
(iv) *B is nondegenerate, i.e., $\mathfrak{g}$ is semisimple.*

Before proving this theorem, we present two important properties of semisimple Lie algebras.

Remark 2.42. If the Lie algebra $\mathfrak{g}$ is the direct sum of noncommutative simple ideals $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$, then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Indeed, on the one hand, $[\mathfrak{g}_i, \mathfrak{g}_j] = \{0\}$ if $i \neq j$, since $\mathfrak{g}_i$ and $\mathfrak{g}_j$ are ideals. On the other hand, $[\mathfrak{g}_i, \mathfrak{g}_i] = \mathfrak{g}_i$, since $\mathfrak{g}_i^{(1)} = [\mathfrak{g}_i, \mathfrak{g}_i]$ is an ideal and $\mathfrak{g}_i$ is a noncommutative simple ideal.

Remark 2.43. If $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ is the direct sum of simple Lie algebras, then this decomposition is unique up to permutations. In fact, consider another decomposition $\mathfrak{g} = \widetilde{\mathfrak{g}}_1 \oplus \cdots \oplus \widetilde{\mathfrak{g}}_m$ and let $\widetilde{X} \in \widetilde{\mathfrak{g}}_k$. Then $\widetilde{X} = \sum_i X_i$, where $X_i \in \mathfrak{g}_i$. Since $\mathfrak{g}_i$ is a noncommutative simple ideal, for each i such that $X_i \neq 0$, there exists $V_i \in \mathfrak{g}_i$ different from X_i, such that $[V_i, X_i] \neq 0$. Hence $[\widetilde{X}, V_i] \neq 0$ is a vector that belongs to both $\mathfrak{g}_i$ and $\widetilde{\mathfrak{g}}_k$. Therefore, the ideal $\mathfrak{g}_i \cap \widetilde{\mathfrak{g}}_k$ is different from $\{0\}$. Since $\mathfrak{g}_i$ and $\widetilde{\mathfrak{g}}_k$ are simple ideals, it follows that $\mathfrak{g}_i = \mathfrak{g}_i \cap \widetilde{\mathfrak{g}}_k = \widetilde{\mathfrak{g}}_k$.

We now prove Theorem 2.41, following Ise and Takeuchi [133, pp. 71–72].

Proof. We proceed by proving the equivalences (i) ⇔ (ii), (i) ⇔ (iii) and (i) ⇔ (iv).

(i) ⇔ (ii). Assume that $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ is the direct sum of noncommutative simple ideals. Let $\pi_i \colon \mathfrak{g} \to \mathfrak{g}_i$ denote the projection onto each factor and note that π_i is a Lie algebra homomorphism. Thus, the projection $\tau_i = \pi_i(\tau)$ is a solvable ideal of $\mathfrak{g}_i$. Since $\mathfrak{g}_i$ is simple, then either τ_i is equal to $\{0\}$ or to $\mathfrak{g}_i$. However, the solvable ideal τ_i cannot be equal to $\mathfrak{g}_i$, since $\mathfrak{g}_i = [\mathfrak{g}_i, \mathfrak{g}_i]$ (see Remark 2.42). Therefore, $\tau_i = \{0\}$ and hence $\tau = \{0\}$.

Conversely, assume that $\tau = \{0\}$. For any ideal $\mathfrak{h}$ of $\mathfrak{g}$, set $\mathfrak{h}^\perp = \{X \in \mathfrak{g} : B(X, \mathfrak{h}) = 0\}$. Note that $\mathfrak{h}^\perp$ and $\mathfrak{h}^\perp \cap \mathfrak{h}$ are ideals, and B restricted to $\mathfrak{h}^\perp \cap \mathfrak{h}$ vanishes identically. It then follows from Proposition 2.39 and the Cartan Theorem 2.40 that $\mathfrak{h}^\perp \cap \mathfrak{h}$ is solvable. Since the radical is trivial, we conclude that $\mathfrak{h}^\perp \cap \mathfrak{h} = \{0\}$. Therefore $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$. Since $\mathfrak{g}$ is finite-dimensional, by induction, $\mathfrak{g}$ is the direct sum of simple ideals. Moreover, $\tau = \{0\}$ implies that each simple ideal is noncommutative.

(i) $\Leftrightarrow$ (iii). Suppose that there exists a nontrivial commutative ideal $\mathfrak{a}$. Then the radical must contain $\mathfrak{a}$, hence it is nontrivial. It then follows from (i) $\Leftrightarrow$ (ii) that $\mathfrak{g}$ is not a direct sum of noncommutative simple ideals.

Conversely, assume that $\mathfrak{g}$ is not a direct sum of noncommutative simple ideals. Then, from (i) $\Leftrightarrow$ (ii), the radical τ is nontrivial, and there exists a positive integer m such that $\tau^{(m-1)} \neq \{0\}$. Set $\mathfrak{a} = \tau^{(m-1)}$ and note that $\mathfrak{a}$ is a nontrivial commutative ideal.

(i) $\Leftrightarrow$ (iv). Assume that $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ is the direct sum of noncommutative simple ideals. From Proposition 2.39, it suffices to prove that $B|_{\mathfrak{g}_i}$ is nondegenerate for each i. Consider a fixed i and let $\mathfrak{h} := \{X \in \mathfrak{g}_i : B(X, \mathfrak{g}_i) = 0\}$. Note that $\mathfrak{h}$ is an ideal of $\mathfrak{g}_i$. Since $\mathfrak{g}_i$ is a simple ideal, either $\mathfrak{h} = \mathfrak{g}_i$ or $\mathfrak{h} = \{0\}$. If $\mathfrak{h} = \mathfrak{g}_i$, then the Cartan Theorem 2.40 implies that $\mathfrak{g}_i$ is solvable, contradicting the fact that $[\mathfrak{g}_i, \mathfrak{g}_i] = \mathfrak{g}_i$. Therefore $\mathfrak{h} = \{0\}$, hence $B|_{\mathfrak{g}_i}$ is nondegenerate.

Conversely, assume that B is nondegenerate. From the last equivalence, it suffices to prove that $\mathfrak{g}$ has no commutative ideals other than $\{0\}$. Let $\mathfrak{a}$ be a commutative ideal of $\mathfrak{g}$. For each $X \in \mathfrak{a}$ and $Y \in \mathfrak{g}$ we have $\operatorname{ad}(X)\operatorname{ad}(Y)(\mathfrak{g}) \subset \mathfrak{a}$. Therefore $B(X,Y) = \operatorname{tr}\left(\operatorname{ad}(X)\operatorname{ad}(Y)\right)|_{\mathfrak{a}}$. On the other hand, since $\mathfrak{a}$ is commutative, $\operatorname{ad}(X)\operatorname{ad}(Y)|_{\mathfrak{a}} = 0$. Therefore $B(X,Y) = 0$ for each $X \in \mathfrak{a}$ and $Y \in \mathfrak{g}$. Since B is nondegenerate, it follows that $\mathfrak{a} = \{0\}$. □

Exercise 2.44. Show that if a compact Lie group G is semisimple, then its center $Z(G)$ is finite. Conclude from Exercise 1.60 that $\mathrm{U}(n)$ is not semisimple.

2.4 Splitting Lie Groups with Bi-invariant Metrics

In Proposition 2.24, we proved that each compact Lie group admits a bi-invariant metric. In this section, we prove that the only simply-connected Lie groups that admit bi-invariant metrics are products of compact Lie groups with vector spaces. We also prove that, if the Lie algebra of a compact Lie group G is simple, then the bi-invariant metric on G is unique up to multiplication by constants.

Theorem 2.45. *Let $\mathfrak{g}$ be a Lie algebra endowed with a bi-invariant metric. Then $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_n$ is the direct orthogonal sum of simple ideals $\mathfrak{g}_i$. In addition, let $\widetilde{G}$ be the connected and simply-connected Lie group with Lie algebra isomorphic to $\mathfrak{g}$. Then $\widetilde{G}$ is isomorphic to the product of normal Lie subgroups $G_1 \times \ldots \times G_n$, such that $G_i = \mathbb{R}$ if $\mathfrak{g}_i$ is commutative, and G_i is compact if $\mathfrak{g}_i$ is noncommutative.*

Proof. In order to verify that $\mathfrak{g}$ is the direct orthogonal sum of simple ideals, it suffices to prove that, if $\mathfrak{h}$ is an ideal, then $\mathfrak{h}^\perp$ is also an ideal. Let $X \in \mathfrak{h}^\perp$, $Y \in \mathfrak{g}$ and $Z \in \mathfrak{h}$. Then, using Proposition 2.26, it follows that

$$Q([X,Y],Z) = -Q([Y,X],Z) = Q(X,[Y,Z]) = 0.$$

Hence $[X,Y] \in \mathfrak{h}^\perp$, and this proves the first assertion.

From Lie's Third Theorem 1.14, given $\mathfrak{g}_i$, there exists a unique connected and simply-connected Lie group G_i with Lie algebra isomorphic to $\mathfrak{g}_i$. In particular, $G_1 \times \dots \times G_n$ is a connected and simply-connected Lie group, with Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_n$. From uniqueness in Lie's Third Theorem, it follows that $\widetilde{G} = G_1 \times \dots \times G_n$. If $\mathfrak{g}_i$ is commutative and simple, then $\mathfrak{g}_i = \mathbb{R}$. Hence, since G_i is connected and simply-connected, $G_i \cong \mathbb{R}$. Else, if $\mathfrak{g}_i$ is noncommutative, observe that there does not exist $X \in \mathfrak{g}_i$, $X \neq 0$, such that $[X,Y] = 0$ for all $Y \in \mathfrak{g}_i$. Indeed, if there existed such X, then $\operatorname{span}\{X\} \subset \mathfrak{g}_i$ would be a nontrivial ideal. From the proof of Theorem 2.35, the Killing form of $\mathfrak{g}_i$ is negative-definite; and from the same theorem, G_i is compact. Finally, since $\mathfrak{g}_i$ is an ideal of $\mathfrak{g}$, it follows from Proposition 2.37 that G_i is a normal subgroup of $\widetilde{G}$. □

The above theorem and Remark 2.42 imply the following corollary.

Corollary 2.46. *Let $\mathfrak{g}$ be a Lie algebra with a bi-invariant metric. Then $\mathfrak{g} = \widetilde{\mathfrak{g}} \oplus Z(\mathfrak{g})$ is a direct sum of ideals, where $\widetilde{\mathfrak{g}}$ is semisimple. In particular, $[\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}] = \widetilde{\mathfrak{g}}$.*

Remark 2.47. Let G be a compact connected Lie group with a nontrivial connected and simply-connected normal Lie subgroup H. From the proof of Theorem 2.45, it follows that there exists a connected normal Lie subgroup L of G such that $G \cong (H \times L)/\Gamma$, where Γ is a finite normal subgroup of $H \times L$. In other words, if G admits a nontrivial normal subgroup, then *up to a finite covering* it splits as a product of Lie groups. For instance, $\mathrm{SU}(n)$ is a normal subgroup of $\mathrm{U}(n)$, and $\mathrm{U}(n) \cong (\mathrm{SU}(n) \times S^1)/\mathbb{Z}_n$, see Exercise 1.60. Furthermore, the Lie algebra of $\mathrm{U}(n)$ splits as a direct sum of ideals $\mathfrak{u}(n) = \mathfrak{su}(n) \oplus Z(\mathfrak{u}(n))$, see Corollary 2.46. From Remark 1.56, another example is $\mathrm{SO}(4) \cong (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathbb{Z}_2$, that has two normal subgroups isomorphic to $\mathrm{SU}(2)$.

Proposition 2.48. *Let G be a compact simple Lie group with Killing form B and bi-invariant metric Q. Then the bi-invariant metric is unique up to rescaling. In addition, (G,Q) is an Einstein manifold, that is, it satisfies (2.6).*

Proof. Let $\widetilde{Q}$ be another bi-invariant metric on G, and let $P\colon \mathfrak{g} \to \mathfrak{g}$ be the positive-definite self-adjoint operator such that $\widetilde{Q}(X,Y) = Q(PX,Y)$. We claim that P and ad commute, that is, $P\,\mathrm{ad}(X) = \mathrm{ad}(X)P$ for all $X \in \mathfrak{g}$. Indeed,

$$\begin{aligned} Q(P\,\mathrm{ad}(X)Y,Z) &= \widetilde{Q}(\mathrm{ad}(X)Y,Z) \\ &= -\widetilde{Q}(Y,\mathrm{ad}(X)Z) \\ &= -Q(PY,\mathrm{ad}(X)Z) \\ &= Q(\mathrm{ad}(X)PY,Z). \end{aligned}$$

Furthermore, eigenspaces of P are $\mathrm{ad}(X)$-invariant, that is, they are ideals. In fact, let $Y \in \mathfrak{g}$ be an eigenvector of P associated to an eigenvalue μ. Then $P\,\mathrm{ad}(X)Y = \mathrm{ad}(X)PY = \mu\,\mathrm{ad}(X)Y$. Since $\mathfrak{g}$ is simple, it follows that $P = \mu\,\mathrm{id}$, hence $\widetilde{Q} = \mu\,Q$.

Since G is compact, it follows from Theorem 2.35 and Corollary 2.33 that $-B$ is a bi-invariant metric. Hence, there exists λ such that $-B(X,Y) = 4\lambda\,Q(X,Y)$. Therefore, from Remark 2.34, $\mathrm{Ric}(X,Y) = -\frac{1}{4}B(X,Y) = \lambda\,Q(X,Y)$, so G is Einstein. □

Exercise 2.49. Let G be a compact semisimple Lie group with Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ given by the direct sum of noncommutative simple ideals. Consider a bi-invariant metric Q on G. Prove that there exist positive numbers λ_j such that

$$Q = \sum_j -\lambda_j B_j,$$

where $B_j = B|_{\mathfrak{g}_j}$ is the restriction of the Killing form to $\mathfrak{g}_j$.

Exercise 2.50. In order to compute the Killing form B of $\mathrm{SU}(n)$, recall that its Lie algebra is

$$\mathfrak{su}(n) = \left\{A \in \mathfrak{gl}(n,\mathbb{C}) : A^* + A = 0,\ \mathrm{tr}\,A = 0\right\},$$

see Exercise 1.51. Consider the following special diagonal matrices in $\mathfrak{su}(n)$,

$$X = \begin{pmatrix} i\theta_1 & & \\ & \ddots & \\ & & i\theta_n \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} i\zeta_1 & & \\ & \ddots & \\ & & i\zeta_n \end{pmatrix}.$$

Use the fact that $\mathrm{tr}\,X = \mathrm{tr}\,Y = 0$, hence $\sum_{i=1}^n \theta_i = \sum_{i=1}^n \zeta_i = 0$, and Exercise 1.38 to verify that computing B in the above X and Y gives:

$$B(X,Y) = \mathrm{tr}\left(\mathrm{ad}(X)\mathrm{ad}(Y)\right) = -2n \sum_{i=1}^n \theta_i\,\zeta_i.$$

From Exercise 2.25, the inner product $Q(Z,W) = \mathrm{Re}\,\mathrm{tr}(ZW^*)$ in $T_e\mathrm{SU}(n)$ can be extended to a bi-invariant metric Q. Since $\mathrm{SU}(n)$ is simple, from Proposition 2.48, there exists a constant $c \in \mathbb{R}$ such that $B = c\,Q$. Conclude that $c = -4n$ and that the Killing form of $\mathrm{SU}(n)$ is

$$B(Z,W) = -2n\,\mathrm{Re}\,\mathrm{tr}(ZW^*), \qquad \text{for all } Z,W \in \mathfrak{su}(n).$$