Chapter 3
Proper and Isometric Actions

In this chapter, we present a concise introduction to the theory of proper and isometric actions. We begin with basic definitions and a quick primer on fiber bundles, leading to the Slice Theorem 3.49 and the Tubular Neighborhood Theorem 3.57. These fundamental results are used throughout the rest of the book; in particular, to establish a strong correspondence between proper and isometric actions in Sect. 3.3. Finally, the stratification of a manifold by orbit types of a proper action is studied in Sect. 3.5.

The following references were a source of inspiration for this chapter, and could serve as further reading material: Bredon [53], Duistermaat and Kolk [79], Kawakubo [137], Onishchik [179], Palais and Terng [182], Pedrosa [11, Part I], Spindola [197] and Walschap [222].

3.1 Proper Actions and Fiber Bundles

In this section, proper actions are introduced together with a preliminary study of fiber bundles, and accompanied by several examples.

Definition 3.1. Let G be a Lie group and M a smooth manifold. A smooth map $\mu\colon G\times M\to M$ is called a *(left) action* of G on M, or a *(left) G-action* on M, if

(i) $\mu(e,x)=x$, for all $x\in M$;
(ii) $\mu(g_1,\mu(g_2,x))=\mu(g_1g_2,x)$, for all $g_1,g_2\in G$ and $x\in M$.

Whenever μ is implicit, the manifold M is called a *G-manifold* or *G-space*, and it is common to denote $\mu(g,x)$ by $g\cdot x$, or even gx, but we avoid this slight abuse of notation until later in the book.

Similarly, one can define a *right action* $\mu\colon M\times G\to M$ of G on M to be a smooth map satisfying properties analogous to (i) and (ii) above, see Example 3.8. For a right action, the short notation for $\mu(x,g)$ is $x\cdot g$, or xg.

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For example, evaluating matrices on vectors defines a left action of $G = \mathrm{GL}(n,\mathbb{R})$ on $M = \mathbb{R}^n$, given by $\mu(A,x) = Ax$. In Chap. 2, we have seen that every Lie group G acts on its Lie algebra $\mathfrak{g}$ via the adjoint action $\mu\colon G \times \mathfrak{g} \to \mathfrak{g}$, $\mu(g,X) = \mathrm{Ad}(g)X$, and this action is studied in detail in Chap. 4. Any Lie subgroup $H \subset G$ of a Lie group G determines a left (and a right) H-action on G, given by multiplication on the left (and, respectively, on the right). In addition, conjugation by elements in H is also an H-action on G.

Definition 3.2. Let $\mu\colon G \times M \to M$ be a left action and $x \in M$. The subgroup

$$G_x := \big\{ g \in G : \mu(g,x) = x \big\} \subset G$$

is called *isotropy group* or *stabilizer* of $x \in M$, and

$$G(x) := \big\{ \mu(g,x) : g \in G \big\} \subset M$$

is called the *orbit* of $x \in M$. If $G(x) = \{x\}$, then x is called a *fixed point* of the action.

The subgroup $\bigcap_{x\in M} G_x$ is called the *ineffective kernel* of the action; if it is the trivial group $\{e\}$, then the action is said to be *effective*. Moreover, if $G_x = \{e\}$, for all $x \in M$, the action is said to be *free*. If for all $x, y \in M$, there exists $g \in G$ with $\mu(g,x) = y$, then the action is said to be *transitive*. Notice that the isotropy group G_x of a fixed point is the entire group G, and an action with fixed points cannot be free.

Remark 3.3. Every G-action can be reduced to an effective action of the quotient of G by the ineffective kernel $\bigcap_{x\in M} G_x$, which is a normal subgroup of G.

Given a (left) G-action on M, let us also define two auxiliary maps[1]:

$$\begin{aligned} \mu^g\colon M &\longrightarrow M & \qquad \mu_x\colon G &\longrightarrow M \\ x &\longmapsto \mu(g,x) & g &\longmapsto \mu(g,x). \end{aligned} \tag{3.1}$$

The key idea of an action is that each $g \in G$ determines a *transformation* of M, namely $\mu^g\colon M \to M$. Since our actions are assumed smooth, μ^g are diffeomorphisms, and hence $\mu^G := \{\mu^g : g \in G\}$ can be identified with a subgroup[2] of the diffeomorphism group $\mathrm{Diff}(M)$. An orbit $G(x)$ consists of all possible images $\mu^g(x)$ for $g \in G$, and the isotropy group G_x consist of all $g \in G$ that fix $x = \mu^g(x)$. If M is endowed with a Riemannian metric g, an action on (M,g) is said to be *isometric*, or *by isometries*, if μ^g is an isometry of (M,g) for all $g \in G$. In this case, the metric g is said to be *G-invariant*, and μ^G can be identified with a subgroup of $\mathrm{Iso}(M,\mathrm{g})$.

[1] Analogous maps can be defined for a right G-action, simply replacing $\mu(g,x)$ by $\mu(x,g)$.

[2] Notice that μ^G is isomorphic to G only if the action μ is effective.

Example 3.4. The following are three basic examples of actions on $M = \mathbb{R}^3$.

(i) Let $G = \mathrm{SO}(3)$ and set $\mu(A,x) = Ax$. The orbit $G(x)$ of a nonzero vector $x \in \mathbb{R}^3$ is the sphere of radius $\|x\|$ centered at the origin, and the isotropy group G_x is isomorphic to the group $\mathrm{SO}(2)$ of rotations around the line spanned by x. The origin $x = 0$ is a fixed point of this action.

(ii) Let $G = \mathrm{SO}(2)$ and set $\mu(A,x) = (A(x_1,x_2),x_3)$. If $x \in \mathbb{R}^3$ is such that $r^2 = x_1^2 + x_2^2 > 0$, then the orbit $G(x)$ is a circle of radius r contained in the plane perpendicular to the z-axis at $(0,0,x_3)$, and the isotropy group is $G_x = \{e\}$. If $r^2 = x_1^2 + x_2^2 = 0$, then x is a fixed point of the action.

(iii) Let $G = \mathrm{SO}(2) \times \mathbb{R}$ and set $\mu((A,b),x) = (A(x_1,x_2), b + x_3)$. The orbit $G(x)$ of a vector $x \in \mathbb{R}^3$ with $r^2 = x_1^2 + x_2^2 > 0$ is a round cylinder of radius r whose rotation axis is the z-axis, and the isotropy group is $G_x = \{e\}$. The orbit $G(x)$ of a vector $x = (0,0,x_3)$ is the z-axis, and the isotropy group $G_x = \mathrm{SO}(2)$ is the group of rotations around the z-axis.

Action (ii) is called a *subaction* of action (i), since it is the restriction of the action (i) to the subgroup of $\mathrm{SO}(3)$ consisting of block diagonal matrices whose first 2×2 block is an element of $\mathrm{SO}(2)$. Action (ii) is also clearly a subaction of action (iii). Both (i) and (ii) are called *linear actions* (or *(linear) representations*, see Definition 1.35) since μ^g is a linear transformation for all $g \in G$. Action (iii) is not linear.

Example 3.5. Let us describe a few operations to construct new actions using other actions. As mentioned in Example 3.4, restricting a G-action on M to a subgroup H defines an H-action on M called a *subaction*. Given (left) actions $\mu_1 \colon G_1 \times M_1 \to M_1$ and $\mu_2 \colon G_2 \times M_2 \to M_2$, we may define a *product action* of $G_1 \times G_2$ on $M_1 \times M_2$ by

$$\begin{aligned} \mu \colon (G_1 \times G_2) \times (M_1 \times M_2) &\to M_1 \times M_2, \\ \mu\big((g_1,g_2),(x_1,x_2)\big) &:= \big(\mu_1(g_1,x_1), \mu_2(g_2,x_2)\big). \end{aligned} \tag{3.2}$$

If $G_1 = G_2 = G$, then restricting the product action (3.2) to the diagonal subgroup $\Delta G := \{(g,g) \in G \times G\}$ defines a subaction called *diagonal action* of G on $M_1 \times M_2$.

Definition 3.6. Consider (left) G-actions $\mu_1 \colon G \times M_1 \to M_1$ and $\mu_2 \colon G \times M_2 \to M_2$. A map $f \colon M_1 \to M_2$ is called *G-equivariant* if $\mu_2(g, f(x)) = f(\mu_1(g,x))$ for all $x \in M_1$ and $g \in G$. Equivariant diffeomorphisms provide a natural notion of equivalence among G-manifolds.

Exercise 3.7. Let $\mu \colon G \times M \to M$ be a G-action. Verify that $\overline{\mu} \colon G \times TM \to TM$ defined by $\overline{\mu}\big(g,(p,v)\big) := \big(\mu(g,p), \mathrm{d}(\mu^g)_p v\big)$ is a G-action on TM, and that the projection $\pi \colon TM \to M$, $\pi(p,v) = p$, is a G-equivariant map. Notice that $\overline{\mu}$ induces a subaction by the isotropy group G_p on T_pM, which is called *isotropy representation*.

Example 3.8. Let $\mu_L \colon G \times M \to M$ be a left G-action, and define $\mu_R \colon M \times G \to M$ by setting $\mu_R(x,g) := \mu_L(g^{-1},x)$. Then μ_R is a right G-action, and the identity map on M is a G-equivariant diffeomorphism with respect to these actions. Right actions can be analogously transformed into left actions.

Exercise 3.9. Let $\mu \colon G \times M \to M$ be a left action. Prove that the isotropy group G_x changes by conjugation as x moves along its orbit $G(x)$. More precisely, show that $G_{\mu(g,x)} = g\,G_x\,g^{-1}$.

If two orbits $G(x)$ and $G(y)$ have nontrivial intersection, then they coincide. This means that orbits of a G-action on M form a partition of M; and hence we can consider the quotient

$$M/G := \big\{G(x) : x \in M\big\},$$

called the *orbit space* or *quotient space* of the G-action on M. The natural projection $\pi \colon M \to M/G$, $\pi(x) := G(x)$, is called the *quotient map*, or *projection map*, and the topology on M/G is determined by declaring that $U \subset M/G$ is open if its preimage $\pi^{-1}(U) \subset M$ is open. This implies that π is continuous, and it is possible to prove that π is also an open map, i.e., maps open subsets of M to open sets of M/G.

Exercise 3.10. Find the orbit spaces M/G of each G-action on $M = \mathbb{R}^3$ from Example 3.4, by identifying a subset of $\mathbb{R}^3$ that contains a point representing each G-orbit.

Example 3.11. Consider the S^1-action on $\mathbb{C}$ by complex multiplication $e^{i\theta} \cdot z = e^{i\theta} z$, and the resulting product action[3] of the torus $S^1 \times S^1$ on $\mathbb{C} \times \mathbb{C}$. The diagonal action

$$\mu \colon S^1 \times (\mathbb{C} \times \mathbb{C}) \to (\mathbb{C} \times \mathbb{C}), \quad \mu\big(e^{i\theta},(z_1,z_2)\big) = \big(e^{i\theta} z_1, e^{i\theta} z_2\big) \tag{3.3}$$

is called the *Hopf action*. This S^1-action on $\mathbb{C}^2$ can be restricted to an S^1-action on the unit sphere $S^3 = \{(z_1,z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$, using the same formula (3.3), which is also called Hopf action.

In order to describe the orbit space S^3/S^1 of the Hopf action, consider the Northern hemisphere $\Sigma := \{(z_1,x_3) \in \mathbb{C} \times \mathbb{R} : |z_1|^2 + x_3^2 = 1, x_3 > 0\}$ of a 2-dimensional subsphere in S^3. If $z = (z_1,z_2) \in S^3$ with $z_2 \neq 0$, then there exists a unique $e^{i\theta} \in S^1$ such that $e^{i\theta} z_2$ is a (real) positive number, i.e., such that $\mu(e^{i\theta},z) \in \Sigma$. On the other hand, the orbit of $z = (z_1,0) \in S^3$ is the equator $\partial\Sigma$. Therefore, *topologically*, $S^3/S^1 = \Sigma \cup [(z_1,0)] = S^2$ is the compactification of the 2-disk, hence a 2-sphere. More details on its geometry are discussed later, see Exercise 3.31.

Exercise 3.12. Let $\theta \colon [0,+\infty) \to \mathbb{R}$ be a smooth function. Verify that $\mu(t,z) := e^{it\theta(|z|)}z$ defines an $\mathbb{R}$-action on $\mathbb{C}$, and that an orbit of this action is either a circle

[3] For more details on this action, see Example 6.48.

centered at the origin or a fixed point. Describe the corresponding orbit space, in terms of the set $\theta^{-1}(0) = \{r \in [0,+\infty) : \theta(r) = 0\}$ where θ vanishes.

In order to study the geometry and topology of orbit spaces, we first need to develop more tools. The next result asserts that, given a smooth action, we can associate to each element of $\mathfrak{g}$ a vector field on M.

Proposition 3.13. *Consider a smooth action $\mu \colon G \times M \to M$.*

(i) *Each $X \in \mathfrak{g}$ induces a smooth vector field X^* on M, called* action field*, by*

$$X^*(p) := \tfrac{\mathrm{d}}{\mathrm{d}t}\mu(\exp(tX), p)\big|_{t=0}.$$

(ii) *The flow of X^* is given by $\varphi_t^{X^*} = \mu^{\exp(tX)}$.*

Proof. Define $\sigma^X \colon M \to TG \times TM$ by $\sigma^X(p) := (X_e, 0_p)$, and note that $\mathrm{d}\mu \circ \sigma^X$ is smooth. Item (i) follows from the fact that $X^*(p)$ is given by

$$\mathrm{d}\mu \circ \sigma^X(p) = \mathrm{d}\mu\left(\tfrac{\mathrm{d}}{\mathrm{d}t}(\exp(tX), p)\big|_{t=0}\right) = \tfrac{\mathrm{d}}{\mathrm{d}t}\mu(\exp(tX), p)\big|_{t=0}.$$

For each $X \in \mathfrak{g}$ and $p \in M$, notice that $\mu^{\exp(tX)}$ satisfies $\mu^{\exp(tX)} = \mathrm{id}$ if $t = 0$, and

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\mu^{\exp(tX)}(p)\big|_{t=t_0} &= \tfrac{\mathrm{d}}{\mathrm{d}s}\mu^{\exp((s+t_0)X)}(p)\big|_{s=0} \\
&= \tfrac{\mathrm{d}}{\mathrm{d}s}\mu\big(\exp(sX)\cdot\exp(t_0X), p)\big)\big|_{s=0} \\
&= X^*(\mu(\exp(t_0X), p)) \\
&= X^*\big(\mu^{\exp(t_0X)}(p)\big),
\end{aligned}$$

proving that $\varphi_t^{X^*} = \mu^{\exp(tX)}$ is the flow of the action field X^*. $\square$

Remark 3.14. It is easy to verify that the association $\mathfrak{g} \ni X \mapsto X^* \in \mathfrak{X}(M)$ defined in (i) is a *Lie anti-homomorphism*, i.e., $[X,Y]^* = -[X^*, Y^*]$.

A consequence of Proposition 3.13 is that, if all orbits of a G-action on M have the same dimension, then the partition

$$\mathscr{F} = \big\{G(p)\big\}_{p \in M} \tag{3.4}$$

is a *foliation* of M, see Definition A.16. The fact that orbits of a G-action are indeed smooth submanifolds is proved in Proposition 3.41, and the fact that for each $v \in T_pG(p)$ there exists $X \in \mathfrak{g}$ such that $X_p^* = v$ follows from (3.11). If the orbits of a G-action have varying dimension, then $\mathscr{F}$ is a *singular* foliation (see Remark 3.42 and Definition 5.1). Foliations of the form (3.4) are called *homogeneous foliations*.

Exercise 3.15. Consider the action $\mu\colon \mathrm{SO}(3)\times\mathbb{R}^3\to\mathbb{R}^3$ in Example 3.4 (i).

(i) Verify that if $X\in\mathfrak{so}(3)\cong\mathbb{R}^3$, then $X^*(p)=X\times p$;
(ii) Consider A_X defined in Exercise 1.9. Verify that e^{tA_X} is a rotation in $\mathbb{R}^3$ about the axis spanned by X with angular speed $\|X\|$.

For future reference, let us state the result of a simple computation regarding the derivative of the map μ_x in (3.1), for both left and right actions.

Proposition 3.16. *If μ is a left action, then the derivative of $\mu_x\colon G\to M$ in (3.1) satisfies* $\ker \mathrm{d}(\mu_x)_{g_0}=T_{g_0}(g_0\,G_x)$. *If μ is a right action, then it satisfies* $\ker \mathrm{d}(\mu_x)_{g_0}=T_{g_0}(G_x\,g_0)$.

We now introduce the concept of *proper actions*, originally due to Palais [180]. Properness is a key hypothesis to extend the theory of actions of compact groups to the noncompact case.

Definition 3.17. An action $\mu\colon G\times M\to M$ is *proper* if the map

$$G\times M\ni(g,x)\longmapsto(\mu(g,x),x)\in M\times M \tag{3.5}$$

is proper, i.e., if the preimage of any compact subset of $M\times M$ under (3.5) is a compact subset of $G\times M$.

It follows directly from the definition that each isotropy group of a proper action is compact, since the preimage of $(x,x)\in M\times M$ under the above map is $G_x\times\{x\}$.

Exercise 3.18. Prove that only proper actions on a compact manifold M are those by compact groups.

The following characterization of proper actions is easy to prove, and implies that all actions by compact groups are proper.

Proposition 3.19. *An action $\mu\colon G\times M\to M$ is proper if and only if for any sequence $\{g_n\}$ in G and any convergent sequence $\{x_n\}$ in M, such that $\{\mu(g_n,x_n)\}$ converges, the sequence $\{g_n\}$ admits a convergent subsequence.*

Exercise 3.20. Let H be a closed subgroup of the Lie group G. Prove that right multiplication $G\times H\ni(g,h)\mapsto g\,h\in G$ is a free proper right H-action.

Exercise 3.21. An action $G\times M\to M$ is called *properly discontinuous* if for all $x\in M$, there exists a neighborhood $U\ni x$ such that $gU\cap U=\emptyset$ for all $g\in G\setminus\{e\}$. Let G be a discrete group that acts on a manifold M. Prove that this action is properly discontinuous if and only if it is free and proper.

Exercise 3.22. Prove that the $\mathbb{R}$-action on $\mathbb{C}$ defined in Exercise 3.12 is not proper if $\theta^{-1}(0)$ is nonempty.

Proper actions are closely related to principal bundles. Before describing this relation, we need some more definitions. We start with that of a *fiber bundle*.

Intuitively, a fiber bundle is a space that *locally* looks like a product space, but *globally* may fail to be a product. In the local scale, such objects are products of a *base* and a *fiber*, where we think of the fiber as a space that is "attached" to each point on the base space to form the fiber bundle. The way in which this attachment is made is locally coherent to make it look like a product at this level, however, it allows certain twists in a global scale. Information about these "twists" is encoded in a Lie group, called the *structure group*. Let us now give the rigorous definition:

Definition 3.23. Let E, B and F be manifolds and G a Lie group. Assume that $\pi\colon E \to B$ a smooth submersion and there is an effective left G-action on F. Furthermore, suppose that B admits an open covering $\{U_\alpha\}$ and that there exist diffeomorphisms $\psi_\alpha\colon U_\alpha \times F \to \pi^{-1}(U_\alpha)$ satisfying:

(i) $\big(\pi \circ \psi_\alpha\big)(b,f) = b$, for all $(b,f) \in U_\alpha \times F$;
(ii) If $U_\alpha \cap U_\beta \neq \emptyset$, then $(\psi_\beta^{-1} \circ \psi_\alpha)(b,f) = (b, \theta_{\alpha,\beta}(b)f)$, where $\theta_{\alpha,\beta}(b) \in G$ and $\theta_{\alpha,\beta}\colon U_\alpha \cap U_\beta \to G$ is smooth.

Then $(E,\pi,B,F,G,\{U_\alpha\},\{\psi_\alpha\})$ is called a *coordinate bundle*. Moreover, $(E,\pi,B,F,G,\{U_\alpha\},\{\psi_\alpha\})$ and $(E,\pi,B,F,G,\{V_\beta\},\{\varphi_\beta\})$ are called *equivalent* if $(\varphi_\beta^{-1} \circ \psi_\alpha)(b,f) = (b, \widetilde{\theta}_{\alpha,\beta}(b)f)$, where $\widetilde{\theta}_{\alpha,\beta}\colon U_\alpha \cap V_\beta \to G$ is smooth. An equivalence class of coordinate bundles, denoted (E,π,B,F,G), is called a *fiber bundle*. In this case, E is called the *total space*, π the *projection*, B the *base*, F the *fiber* and G the *structure group*. For each $b \in B$, the preimage $\pi^{-1}(b)$ is called the *fiber over b* and is often denoted E_b. Furthermore, ψ_α are called *bundle charts* and $\theta_{\alpha,\beta}$ *transition functions*. A fiber bundle is usually denoted by $F \to E \to B$, or simply by its total space E, if the underlying structure is implicitly understood. We also say that E is an F-bundle over B.

Example 3.24. The product manifold $E = B \times F$ is the total space of the *trivial bundle*, for which $\pi\colon E \to B$ is the projection onto the first factor, $G = \{e\}$ and the bundle charts are the identity.

Remark 3.25. The total space E of a fiber bundle can be reconstructed by *gluing* trivial products $U_\alpha \times F$ according to transition functions $\theta_{\alpha,\beta}$. More precisely, consider the disjoint union $\bigsqcup_\alpha (U_\alpha \times F)$. Whenever $x \in U_\alpha \cap U_\beta$, identify $(x,f) \in U_\alpha \times F$ with $(x, \theta_{\alpha,\beta}(x)(f)) \in U_\beta \times F$. Then the quotient space $E = \bigsqcup_\alpha (U_\alpha \times F)/\sim$ and the projection onto the first factor is the bundle projection.

Examples of fiber bundles were already encountered in the previous chapters, such as the tangent bundle TM of a manifold M. A point in TM consists of a pair (p,v), where $v \in T_pM$. The projection $\pi\colon TM \to M$ is the map $(p,v) \mapsto p$, and the fiber over $p \in M$ is $\pi^{-1}(p) = T_pM$. The tangent bundle of a manifold is a particular example of a special class of fiber bundles called *vector bundles*. Vector bundles are, by definition, fiber bundles whose fiber is a vector space V and whose structure group is the group $\mathrm{GL}(V)$ of automorphisms of V. The dimension of V is called the *rank* of the vector bundle. Another example of vector bundle is the *cotangent bundle* TM^* of M, whose fiber at each $p \in M$ is the dual space T_pM^*. Verifying that these are indeed vector bundles of rank $\dim M$ is an easy exercise.

Another important example of vector bundle is the *normal bundle* of a submanifold. Let N be a submanifold of M, and consider their tangent bundles TN and TM, respectively. There is a natural inclusion of T_pN in T_pM for all $p \in N$, and hence one can consider the quotient space $\nu_p(N) := T_pM/T_pN$. This is the fiber of the normal bundle of N over p. In other words, the normal bundle of N is defined as

$$\nu(N) := \bigsqcup_{p\in N} \nu_p(N) = \bigsqcup_{p\in N} T_pM/T_pN, \tag{3.6}$$

and is a vector bundle over N with rank equal to the codimension of N in M. Notice that the above definition does not require a Riemannian metric, i.e., in this context, there is no canonical way of measuring the *angle* between the normal space ν_pN and the tangent space T_pN. However, if we consider M endowed with a Riemannian metric g, then the above definition is equivalent to

$$\nu(N) := \bigsqcup_{p\in N} \left\{ v \in T_pM : \mathrm{g}_p(v,w) = 0 \text{ for all } w \in T_pN \right\},$$

which explains why it is called the *normal* bundle. For the same reason, the normal bundle $\nu(N)$ is sometimes denoted $TN^{\perp}$.

Remark 3.26. Certain geometric structures on a fiber bundle can be encoded as a *reduction* of its structural group G to a Lie subgroup $H \subset G$. This corresponds to the bundle admitting an atlas with H-valued transition functions. For instance, if M is a manifold with $\dim M = n$, the presence of a Riemannian metric on M allows to reduce the structural group $\mathrm{GL}(n,\mathbb{R})$ of TM to the subgroup $\mathrm{O}(n)$, and furthermore to $\mathrm{SO}(n)$, provided M is orientable.

Definition 3.27. A *smooth section* of a fiber bundle $F \to E \to B$ is a smooth map $\sigma\colon B \to E$ with $\pi \circ \sigma = \mathrm{id}_B$, that is, σ is a smooth choice of $\sigma(b) \in E_b$ in each fiber.

Example 3.28. Smooth functions on M are sections of the trivial bundle $M \times \mathbb{R}$. Vector fields are sections of the tangent bundle TM, and differential 1-forms are sections of the cotangent bundle TM^*. If N is a submanifold of M, then sections of the normal bundle $\nu(N)$ are normal vector fields along N.

A fiber bundle (P,ρ,B,F,G) is called a *principal G-bundle*, or *principal bundle*, if $F = G$ and the action of G on itself is by left translations. As explained in Proposition 3.33 and Theorem 3.34 below, a fiber bundle $G \to P \to B$ is principal if and only if its fibers are orbits of a free proper G-action (whose orbit space is hence B).

Example 3.29. There are two very familiar examples of principal G-bundles, related to basic structures of a smooth manifold M. First, the universal covering $\rho\colon \widetilde{M} \to M$ is a principal G-bundle over M, where $G = \pi_1(M)$ is its fundamental group. Notice that $\pi_1(M)$ determines a properly discontinuous action on $\widetilde{M}$ by deck transformations, see Exercise 3.21. Second, consider the *frame bundle* of M, defined by

$$B(TM) := \bigsqcup_{x\in M} B(T_xM),$$

where $B(T_xM)$ is the set of all frames (i.e., ordered bases) on the vector space T_xM. Then $B(TM)$ is a principal G-bundle over M, with $G = \mathrm{GL}(n,\mathbb{R})$, where $n = \dim M$. Indeed, note that $B(T_xM)$ is diffeomorphic to $\mathrm{GL}(n,\mathbb{R})$, since each frame $\xi = \{\xi_i\}$ in $B(T_xM)$ determines a linear isomorphism $\xi\colon \mathbb{R}^n \to T_xM$ given by $\xi(e_i) = \xi_i$, where $\{e_i\}$ is the canonical basis of $\mathbb{R}^n$. Note also that composition of isomorphisms $\mu(\xi,A) := \xi\circ A$ determines a transitive right action of $\mathrm{GL}(n,\mathbb{R})$ on $B(T_xM)$.

Example 3.30. The *Hopf bundles* $S^1 \to S^{2n+1} \to \mathbb{C}P^n$ and $S^3 \to S^{4n+3} \to \mathbb{H}P^n$ are respectively principal S^1-bundles and S^3-bundles. Recall that the complex projective space $\mathbb{C}P^n := S^{2n+1}/S^1$ and the quaternionic projective space $\mathbb{H}P^n := S^{4n+3}/S^3$ are, respectively, the spaces of complex lines in $\mathbb{C}^{n+1}$ and quaternionic lines in $\mathbb{H}^{n+1}$. These are parametrized by equivalence classes $[v]$ of unit vectors $v\in S^{2n+1}\subset\mathbb{C}^{n+1}$, or $v\in S^{4n+3}\subset\mathbb{H}^{n+1}$, that span the same complex or quaternionic line, respectively. Such an equivalence class $[v]$ is precisely the orbit of the unit vector v under the multiplication action of the group S^1 of unit complex numbers or the group $S^3\cong \mathrm{Sp}(1)$ of unit quaternions. For more details, see Example 6.10.

Exercise 3.31 ($\star$). In this exercise, we study the Hopf bundle $S^1\to S^3\to\mathbb{C}P^1$, see also Example 3.11. Recall that $\mathbb{C}P^1$ is isometric to the sphere $S^2\left(\frac12\right)\subset\mathbb{R}^3$ of radius $\frac12$. Write $S^3 = \{(z_1,z_2)\in\mathbb{C}^2 : |z_1|^2+|z_2|^2 = 1\}$ as a subset of $\mathbb{C}^2$ and $S^2\left(\frac12\right) = \{(z,x)\in\mathbb{C}\times\mathbb{R} : |z|^2+x^2=\frac14\}$ as a subset of $\mathbb{C}\times\mathbb{R}$. Consider the map

$$\pi\colon S^3\to S^2\left(\tfrac12\right),\quad \pi(z_1,z_2) := \left(z_1\,\overline{z_2}, \frac{|z_1|^2-|z_2|^2}{2}\right). \tag{3.7}$$

(i) Prove that π is well-defined, i.e., it maps S^3 to $S^2\left(\frac12\right)$;
(ii) Verify that $\pi(z_1,z_2)=\pi(w_1,w_2)$ if and only if there exists $e^{i\theta}\in S^1$ such that $(w_1,w_2) = (e^{i\theta}z_1, e^{i\theta}z_2)$;
(iii) Let $N = \left(0,\frac12\right)\in S^2\left(\frac12\right)$ be the North pole, and consider the open subsets $U_\pm = S^2\left(\frac12\right)\setminus\{\pm N\}$. Verify that $\pi^{-1}(U_\pm) = S^3\setminus\pi^{-1}(\pm N)$;
(iv) Describe bundle charts $\psi_\pm\colon U_\pm\times S^1\to\pi^{-1}(U_\pm)$, to verify that $\pi\colon S^3\to S^2\left(\frac12\right)$ is a fiber bundle, and moreover, a principal S^1-bundle. Observe that π is also a Riemannian submersion, see also Exercise 3.81.

Remark 3.32. It is possible to prove that a principal bundle admits a smooth section if and only if it is trivial, see Walschap [222, p. 68]. This follows from the correspondence between sections and equivariant bundle charts on a principal bundle. In particular, the Hopf bundles in Example 3.30 do not admit any smooth sections.

Proposition 3.33. *Principal G-bundles $G\to P\to B$ have an underlying free proper right G-action $\mu\colon P\times G\to P$ whose orbits are the fibers of the bundle.*

Proof. Denote by $\rho\colon P\to B$ the projection map. Given $x\in M$, let $\psi_\alpha\colon U_\alpha\times G\to \rho^{-1}(U_\alpha)$ be a bundle chart with $x=\psi_\alpha(b,f)\in\rho^{-1}(U_\alpha)$, and set

$$\mu(x,g):=\psi_\alpha\big(\psi_\alpha^{-1}(x)\cdot g\big), \tag{3.8}$$

where $(b,f)\cdot g:=(b,f\cdot g)$, for all $b\in B$ and $f,g\in G$. Let us verify that this action is well-defined, i.e., that the above definition does not depend on the choice of ψ_α. This follows from the fact that the structure group acts on the *left*, while (3.8) is a *right* action (see Exercise 3.40). More precisely, for any $x\in\rho^{-1}(U_\alpha)\cap\rho^{-1}(U_\beta)$,

$$\begin{aligned}\psi_\alpha\big(\psi_\alpha^{-1}(x)\cdot g\big)&=\psi_\alpha(b,fg)\\&=\psi_\beta(b,\theta_{\alpha,\beta}(b)fg)\\&=\psi_\beta\big((b,\theta_{\alpha,\beta}(b)f)\cdot g\big)\\&=\psi_\beta\big(\psi_\beta^{-1}(x)\cdot g\big).\end{aligned}$$

The definition of μ and Exercise 3.20 imply that μ is a free proper right action. The fact that its orbits coincide with the fibers is immediate from (3.8). □

The next theorem provides a converse to the above result, and a method to build principal bundles. It also shows that, in special cases, M/G is a smooth manifold.

Theorem 3.34. *Let $\mu\colon M\times G\to M$ be a free proper right action. Then the orbit space M/G admits a smooth structure such that $G\to M\to M/G$ is a principal G-bundle, where the bundle projection map $\rho\colon M\to M/G$ is the quotient map.*

Remark 3.35. The smooth structure on M/G is such that $\rho\colon M\to M/G$ is smooth, and a map $h\colon M/G\to N$ is smooth if and only if $h\circ\rho$ is smooth. These properties uniquely characterize the smooth structure of M/G.

Proof (Sketch). Let S be a submanifold containing x, with[4] $T_xM=T_xS\oplus \mathrm{d}(\mu_x)_e(\mathfrak{g})$, and let $\varphi\colon S\times G\to M$ be the restriction of the G-action to $S\times G$, that is, $\varphi:=\mu|_{S\times G}$.

Claim 3.36. Up to shrinking S, there exists a G-invariant neighborhood U of $G(x)$ such that $\varphi\colon S\times G\to U$ is a G-equivariant diffeomorphism, where the G-action on $S\times G$ is by right multiplication on the second factor.

In order to prove this claim, we first verify that $\mathrm{d}\varphi_{(s,e)}$ is a linear isomorphism and hence, by the Inverse Function Theorem, φ is a local diffeomorphism on a neighborhood of $S\times\{e\}$, up to shrinking S. Indeed, note that $\varphi(\cdot,e)$ is the identity map on S, and the linearization of $\varphi(s,\cdot)$ at e also has trivial kernel, since the action is free, see Proposition 3.16. It then follows from

$$\mathrm{d}\varphi_{(s,g)}(X,\mathrm{d}R_gY)=\big(\mathrm{d}(\mu^g)_s\circ\mathrm{d}\varphi_{(s,e)}\big)(X,Y)$$

[4] A posteriori, this condition means that the submanifold S is transverse to the orbit $G(x)$.

that φ is a local diffeomorphism near each point of $S \times G$. Using that the action is proper and Proposition 3.19, a standard argument with sequences implies that φ must be injective, which completes the proof of Claim 3.36. Note that φ is not a chart for M/G, since S is contained in M and not in M/G.

The next part of the proof is to identify S with an open subset of M/G. To this aim, we recall that the quotient map $\rho\colon M \to M/G$ is continuous and open. Moreover, using that the action is proper, it is possible to prove that M/G is Hausdorff.[5] The above considerations and Claim 3.36 imply that $\rho(S)$ is an open subset of M/G and $\rho|_S\colon S \to \rho(S)$ is a homeomorphism. This allows to define charts on M/G, and bundle charts on the bundle $G \to M \to M/G$. The compatibility of these charts and bundle charts follows from the following.

Claim 3.37. Let S_1 and S_2 be submanifolds transverse to two different orbits, with $W := \rho(S_1) \cap \rho(S_2) \neq \emptyset$. Set $\rho_i := \rho|_{S_i}\colon S_i \to \rho(S_i)$, $V_i := \rho_i^{-1}(W)$ and $\varphi_i := \mu|_{S_i \times G}$.

(i) $(\rho_1^{-1} \circ \rho_2)\colon V_2 \to V_1$ is a diffeomorphism;
(ii) $(\psi_2^{-1} \circ \psi_1)(b,g) = (b, \theta(b)g)$, where $\psi_i\colon \rho(S_i) \times G \to \rho^{-1}\big(\rho(S_i)\big)$ is given by $\psi_i(b,g) = \varphi_i\big(\rho_i^{-1}(b), g\big)$ and $b \in W$.

In order to prove (i), note that $\varphi_2^{-1}(V_1)$ is a graph, i.e., $\varphi_2^{-1}(V_1) = \{(s, \widetilde{\theta}(s)) : s \in V_2\}$, which implies that $\widetilde{\theta}\colon S \to G$ is smooth. Thus, $(\rho_1^{-1} \circ \rho_2)(s) = \varphi(s, \widetilde{\theta}(s))$ is smooth, and also a diffeomorphism. As for (ii), defining the transition function as $\theta(b) := \widetilde{\theta}(\rho_2^{-1}(b))$ guarantees that ψ_1 and ψ_2 satisfy (ii) in Definition 3.23. $\square$

Corollary 3.38. *Let G be a Lie group and H be a closed subgroup. Consider the right H-action on G by multiplication on the right. The corresponding orbit space G/H is a manifold, and the quotient map $\rho\colon G \to G/H$, $\rho(g) = gH$, determines a principal H-bundle $H \to G \to G/H$. In addition, if H is a normal subgroup of G, then G/H is a Lie group and ρ is a Lie group homomorphism.*

Proof. The first part of the above statement follows directly from Theorem 3.34 and Exercise 3.20. To show that G/H is a Lie group if H is a normal subgroup, define

$$\begin{aligned} \alpha\colon G \times G \ni (g_1, g_2) &\longmapsto g_1 g_2^{-1} \in G \\ \widetilde{\alpha}\colon G/H \times G/H \ni (g_1 H, g_2 H) &\longmapsto g_1 g_2^{-1} H \in G/H. \end{aligned} \tag{3.9}$$

Note that, since H is normal, $\widetilde{\alpha}$ is well-defined. The fact that $\rho\colon G \to G/H$ is a projection of a bundle and $\rho \circ \alpha = \widetilde{\alpha} \circ (\rho \times \rho)$ imply that $\widetilde{\alpha}$ is smooth. Therefore G/H is a Lie group, see Remark 1.2. $\square$

[5] In fact, since (3.5) is a proper map between locally compact Hausdorff spaces, it is also *closed*, i.e., maps closed subsets to closed subsets. Thus, its image $\mathscr{R} = \{(x,y) \in M \times M : G(x) = G(y)\}$ is closed. As $\rho\colon M \to M/G$ is an open map, $\rho(M \times M \setminus \mathscr{R})$ is open. This is easily seen to be the complement of the diagonal in $M/G \times M/G$, which is hence closed, proving that M/G is Hausdorff.

Example 3.39. Consider the free right action of a Lie group G on itself by multiplication on the right (see Exercise 3.20) and the induced diagonal action on $G \times G$,

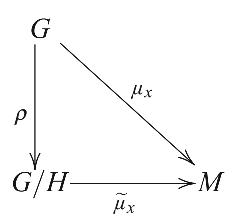
$$\mu\colon (G\times G)\times \Delta G \to G\times G, \quad \mu\big((g_1,g_2),(g,g)\big) := \big(g_1 g, g_2 g\big), \tag{3.10}$$

see Example 3.5. This action is clearly free, hence by Theorem 3.34, there is a principal bundle $\Delta G \to G\times G \to (G\times G)/\Delta G$. The map $\alpha\colon G\times G\to G$ defined in (3.9) descends to a map $\overline{\alpha}\colon (G\times G)/\Delta G \to G$, $\overline{\alpha}\big([g_1,g_2]\big) = g_1 g_2^{-1}$, which is well-defined since $\alpha(\Delta G) = \{e\}$. It is easy to verify that $\overline{\alpha}$ is determines a Lie group isomorphism $(G\times G)/\Delta G \cong G$. This provides a convenient way of rewriting G as an orbit space, which is useful in several applications (see Remark 4.12).

Exercise 3.40. Suppose that M is a manifold with a free proper right G_1-action $\mu_1\colon M\times G_1 \to M$ and a left G_2-action $\mu_2\colon G_2\times M\to M$. Suppose that μ_1 and μ_2 are actions that *commute*,[6] that is, $\mu_1(\mu_2(g_2,x),g_1) = \mu_2(g_2,\mu_1(x,g_1))$ for all $x\in M$ and $g_i \in G_i$. Show that μ_2 descends to a left G_2-action $\widetilde{\mu}_2\colon G_2\times M/G_1 \to M/G_1$, given by $\widetilde{\mu}_2(g_2,G_1(x)) := G_1(\mu_2(g_2,x))$, and verify that $\pi\colon M\to M/G_1$ is G_2-equivariant.

Each isotropy group $H = G_x$ of a G-action on M is such that G/H is a smooth manifold, by Corollary 3.38. We now prove that the orbit $G(x)$ is the image of an immersion of G/H into M.

Proposition 3.41. *Let $\mu\colon G\times M\to M$ be a left action and define $\widetilde{\mu}_x\colon G/H\to M$ by $\widetilde{\mu}_x\circ\rho = \mu_x$, where $H = G_x$ is the isotropy at $x\in M$ and $\rho\colon G\to G/H$ is the quotient map. Then $\widetilde{\mu}_x$ is a G-equivariant*[7] *injective immersion, with image $G(x)$. In particular, $G(x)$ is an immersed submanifold of M whose tangent space at $x\in M$ is $T_xG(x) = \mathrm{d}(\mu_x)_e(\mathfrak{g})$. In addition, if the action is proper, then $\widetilde{\mu}_x$ is an embedding and $G(x)$ is an embedded submanifold of M.*



Proof. According to Corollary 3.38, $H\to G\to G/H$ is a principal H-bundle. This implies that $\widetilde{\mu}_x$ is smooth. It follows from Proposition 3.16 that the derivative of $\widetilde{\mu}_x$ at every point is injective, and hence $\widetilde{\mu}_x$ is an injective immersion. The fact that $\widetilde{\mu}_x$ is an embedding when the action is proper can be proved using Proposition 3.19. □

[6] More generally, two (either left or right) actions μ_1 and μ_2 on M are said to *commute* if the induced transformations (3.1) on M satisfy $(\mu_1)^{g_1}\circ(\mu_2)^{g_2} = (\mu_2)^{g_2}\circ(\mu_1)^{g_1}$ for all $g_i\in G_i$.

[7] The G-action on G/H is by left translations, that is, $\overline{g}\cdot gH := \overline{g}gH$, see Exercise 3.40 and (6.13).

As observed above, the tangent space $T_xG(x)$ to an orbit is the image of the linear map $\mathrm{d}(\mu_x)_e\colon \mathfrak{g} \to T_xM$. Denote by $\mathfrak{g}_x$ the Lie algebra of the isotropy group G_x, and let $\mathfrak{m}_x$ be a complement, i.e., $\mathfrak{g} = \mathfrak{g}_x \oplus \mathfrak{m}_x$ as vector spaces. Then, since $\ker \mathrm{d}(\mu_x)_e = \mathfrak{g}_x$, we have that $\mathfrak{m}_x$ can be identified with $T_xG(x)$. More precisely, there is a natural identification

$$\mathfrak{m}_x \ni X \longmapsto X^*(x) \in T_xG(x), \tag{3.11}$$

where X^* is the action field on M induced by $X \in \mathfrak{m}_x$, see Proposition 3.13.

Remark 3.42. From Proposition 3.41, orbits $G(x)$ of a left action $\mu\colon G \times M \to M$ are immersed submanifolds of M. The dimension of these submanifolds $G(x)$ is *lower semi-continuous* on x, i.e., for each $x_0 \in M$, the dimension of orbits $G(x)$ for x near x_0 is greater than or equal to $\dim G(x_0)$. This can be proved using that $\dim G(x)$ is the rank of the linear map $\mathrm{d}(\mu_x)_e\colon \mathfrak{g} \to T_xM$, which is in fact constant along $G(x)$. From continuity of $x \mapsto \mathrm{d}(\mu_x)_e$ and lower semi-continuity of the rank of a continuous family of linear maps, it follows that $\dim G(x)$ is lower semi-continuous.

Example 3.43. Consider the 2-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ as the orbit space of the right $\mathbb{Z}^2$-action by translations, $\mu_1\colon \mathbb{Z}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, $\mu_1\big((n,m),(x,y)\big) = (x+n, y+m)$, recall 1.14. Choose a real number $\alpha \in \mathbb{R}$ and define a left $\mathbb{R}$-action on $\mathbb{R}^2$ by $\mu_2(t,(x,y)) := (x+t, y+\alpha t)$. By Exercise 3.40, μ_2 descends to an $\mathbb{R}$-action $\widetilde{\mu}_2$ on T^2. If $\alpha \in \mathbb{Q}$ is rational, then the orbits of $\widetilde{\mu}_2$ are embedded circles in T^2, which correspond to straight line segments in $\mathbb{R}^2$ joining two points on the lattice $\mathbb{Z}^2$. On the other hand, if $\alpha \notin \mathbb{Q}$ is irrational, then the orbits of $\widetilde{\mu}_2$ are *dense*[8] (and uniformly distributed) in T^2; in particular, they are not embedded submanifolds. Recall that, by Exercise 3.18, the $\mathbb{R}$-action $\widetilde{\mu}_2$ on T^2 cannot be proper, despite μ_2 being proper. Analogous statements can be proved for higher dimensional tori.

Exercise 3.44. Verify that the following are equivariantly diffeomorphic:

(i) $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$;
(ii) $\mathbb{R}P^n = \mathrm{SO}(n+1)/S(\mathrm{O}(n) \times \mathrm{O}(1))$;
(iii) $\mathbb{C}P^n = \mathrm{SU}(n+1)/S(\mathrm{U}(n) \times \mathrm{U}(1))$;
(iv) $\mathbb{H}P^n = \mathrm{Sp}(n+1)/\mathrm{Sp}(n) \times \mathrm{Sp}(1)$.

Here, $S(\mathrm{O}(n) \times \mathrm{O}(1))$ is the subgroup of $\mathrm{SO}(n+1)$ formed by the matrices

$$A = \begin{pmatrix} B & 0 \\ 0 & \pm 1 \end{pmatrix},$$

where $B \in \mathrm{O}(n)$ and $\det A = 1$, and analogously for $S(\mathrm{U}(1) \times \mathrm{U}(n))$.

[8] This is a consequence of the so-called Kronecker Approximation Theorem.

Hint: Use Example 3.30 and Proposition 3.41. To prove (ii), note that the $\mathrm{SO}(n+1)$-action on S^n commutes with the antipodal $\mathbb{Z}_2$-action and hence descends to an action on $\mathbb{R}P^n$, see Exercise 3.40. More details on the solution to this exercise are given in Example 6.10.

Remark 3.45. The same idea in Exercise 3.44 can be used to find equivariant diffeomorphisms between the k-Grassmannians on $\mathbb{R}^n$, $\mathbb{C}^n$ and $\mathbb{H}^n$ and quotients of classical Lie groups. Recall that if V is a finite-dimensional vector space, the *k-Grassmannian* on V is defined as

$$\mathrm{Gr}_k(V) := \big\{W \text{ linear subspace of } V : \dim W = k\big\}.$$

In particular, $\mathrm{Gr}_1(\mathbb{R}^{n+1}) = \mathbb{R}P^n$, $\mathrm{Gr}_1(\mathbb{C}^{n+1}) = \mathbb{C}P^n$ and $\mathrm{Gr}_1(\mathbb{H}^{n+1}) = \mathbb{H}P^n$. Observe that $\mathrm{SO}(n)$, $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$ act on, respectively, $\mathrm{Gr}_k(\mathbb{R}^n)$, $\mathrm{Gr}_k(\mathbb{C}^n)$ and $\mathrm{Gr}_k(\mathbb{H}^n)$. These transitive actions provide the equivariant diffeomorphisms:

(ii') $\mathrm{Gr}_k(\mathbb{R}^n) = \mathrm{SO}(n)/S(\mathrm{O}(n-k)\times \mathrm{O}(k))$;
(iii') $\mathrm{Gr}_k(\mathbb{C}^n) = \mathrm{SU}(n)/S(\mathrm{U}(n-k)\times \mathrm{U}(k))$;
(iv') $\mathrm{Gr}_k(\mathbb{H}^n) = \mathrm{Sp}(n)/\mathrm{Sp}(n-k)\times \mathrm{Sp}(k)$.

Remark 3.46. The above also be used to prove that $\mathrm{SU}(n)/T$ is a *complex flag manifold*, where T is the subgroup of $\mathrm{SU}(n)$ formed by diagonal matrices. Indeed, let $F_{1,\dots,n-1}(\mathbb{C}^n)$ be the set of *complex flags* on $\mathbb{C}^n$, i.e.,

$$F_{1,\dots,n-1}(\mathbb{C}^n) := \big\{\{0\}\subset E_1 \subset \ldots \subset E_{n-1} : E_i \in \mathrm{Gr}_i(\mathbb{C}^n)\big\}.$$

Set $\widetilde{F} := \{(l_1,\dots,l_n) : l_1\dots l_n \text{ orthogonal lines of } \mathbb{C}^n\}$. On the one hand, there is a natural bijection between $\widetilde{F}$ and $F_{1,\dots,n-1}(\mathbb{C}^n)$ given by $(l_1,\dots,l_n)\mapsto \{0\}\subset E_1 \subset \ldots \subset E_{n-1}$, where $E_i = l_1\oplus\cdots\oplus l_i$. On the other hand, $\mathrm{SU}(n)$ acts naturally on $\widetilde{F}$ and this action is transitive. Note that, for an orthogonal basis $\{e_1,\dots,e_n\}$ of $\mathbb{C}^n$, the isotropy group of $(\mathbb{C}e_1,\dots,\mathbb{C}e_n)\in\widetilde{F}$ is T. Therefore, $\mathrm{SU}(n)/T = \widetilde{F} = F_{1,\dots,n-1}(\mathbb{C}^n)$.

3.2 Slices and Tubular Neighborhoods

In this section, we discuss the construction of associated bundles to a principal bundle, and prove two foundational results for the theory discussed in the remainder of the book: the Slice Theorem 3.49 and the Tubular Neighborhood Theorem 3.57.

Definition 3.47. Let $\mu\colon G\times M\to M$ be an action. A *slice* at $x_0\in M$ is an embedded submanifold S_{x_0} containing x_0 and satisfying the following properties:

(i) $T_{x_0}M = \mathrm{d}\mu_{x_0}\mathfrak{g}\oplus T_{x_0}S_{x_0}$ and $T_xM = \mathrm{d}\mu_x\mathfrak{g} + T_xS_{x_0}$, for all $x\in S_{x_0}$;
(ii) S_{x_0} is invariant under G_{x_0}, i.e., if $x\in S_{x_0}$ and $g\in G_{x_0}$, then $\mu(g,x)\in S_{x_0}$;
(iii) If $x\in S_{x_0}$ and $g\in G$ are such that $\mu(g,x)\in S_{x_0}$, then $g\in G_{x_0}$.

Example 3.48. Consider the action of $\mathrm{SO}(2)\times\mathbb{R}$ on $\mathbb{R}^3$ defined in Example 3.4 (iii). A slice for this action at the point $x=(x_1,x_2,x_3)\in\mathbb{R}^3$ is either a straight line segment $S_x = \{(t\,x_1, t\,x_2, x_3) : |t-1| < \varepsilon\}$ if $r^2 = x_1^2 + x_2^2 > 0$, or else, a disk $S_x = \{(y_1, y_2, x_3), \in \mathbb{R}^3 : y_1^2 + y_2^2 < \varepsilon\}$ if $r^2 = x_1^2 + x_2^2 = 0$. Notice that these slices are invariant under the corresponding isotropies $G_x = \{e\}$ and $G_x = \mathrm{SO}(2)$, respectively.

Slice Theorem 3.49. *Let $\mu\colon G \times M \to M$ be a proper action and $x_0 \in M$. Then there exists a slice S_{x_0} at x_0.*

Proof. We start by constructing a Riemannian metric on M for which the subaction of the isotropy group $H = G_{x_0}$ is by isometries. This metric is defined by averaging the H-action, that is,

$$\langle X, Y \rangle_p := \int_H \langle\langle \mathrm{d}\mu^g X, \mathrm{d}\mu^g Y \rangle\rangle_{\mu(g,p)}\ \omega, \tag{3.12}$$

where ω is a right-invariant volume form on H and $\langle\langle \cdot, \cdot \rangle\rangle$ is an arbitrary Riemannian metric on M. Arguments similar to those in the proof of Proposition 2.24 imply that H acts by isometries on M equipped with the metric $\langle \cdot, \cdot \rangle$.

It is easy to see that, if $g \in H$, then $\mathrm{d}\mu^g\colon T_{x_0}M \to T_{x_0}M$ leaves invariant the tangent space $T_{x_0}G(x_0)$. Since H acts by isometries, we have that $\mathrm{d}\mu^g$ also leaves invariant the orthogonal complement to $T_{x_0}G(x_0)$ in $T_{x_0}M$, which is the normal space $\nu_{x_0}G(x_0)$. Define a candidate to slice at x_0 by setting

$$S_{x_0} := \exp_{x_0}\big(B_\varepsilon(0)\big), \tag{3.13}$$

where $B_\varepsilon(0)$ is an open ball of radius $\varepsilon > 0$ around the origin in the normal space $\nu_{x_0}G(x_0)$. Since $\mathrm{d}\mu^g$ leaves invariant $\nu_{x_0}G(x_0)$ and isometries map geodesics to geodesics, S_{x_0} is invariant under the H-action. Therefore, item (ii) of Definition 3.47 is satisfied. Item (i) is satisfied by the construction of S_{x_0} and continuity of $\mathrm{d}\mu$.

It only remains to verify item (iii), which we prove by contradiction. Suppose (iii) is not satisfied for any $\varepsilon > 0$ in (3.13). Then there exists a sequence $\{x_n\}$ in S_{x_0} and a sequence $\{g_n\}$ in G such that $\lim x_n = x_0$, $\lim \mu(g_n, x_n) = x_0$, $\mu(g_n, x_n) \in S_{x_0}$ and $g_n \notin H$. Since the action is proper, there exists a subsequence, which we also denote $\{g_n\}$, that converges to $g \in H$. Set $\widetilde{g}_n := g^{-1} g_n$. Then $\lim \widetilde{g}_n = e$, $\lim \mu(\widetilde{g}_n, x_n) = x_0$, $\mu(\widetilde{g}_n, x_n) \in S_{x_0}$ and $\widetilde{g}_n \notin H$. Using Proposition 3.16 and the Inverse Function Theorem, one can prove the next claim, up to possibly shrinking S_{x_0}.

Claim 3.50. There exists a submanifold $C \subset G$ containing e, such that $\mathfrak{g} = \mathfrak{h} \oplus T_eC$, where $\mathfrak{h}$ is the Lie algebra of H. Furthermore, $\varphi\colon C \times S_{x_0} \ni (c, s) \mapsto \mu(c, s) \in M$ is a diffeomorphism onto its image.

It follows from the Inverse Function Theorem that for each $\widetilde{g}_n$, there exists a unique $c_n \in C$ and $h_n \in H$, such that $\widetilde{g}_n = c_n h_n$. Since $\widetilde{g}_n \notin H$, we conclude that $c_n \neq e$. On the other hand, $\mu(h_n, x_n) \in S_{x_0}$ because $h_n \in H$. The fact that $c_n \neq e$ and Claim 3.50 imply that $\mu(c_n, \mu(h_n, x_n)) \notin S_{x_0}$. This is a contradicts the fact that $\mu(\widetilde{g}_n, x_n) \in S_{x_0}$, proving that (iii) must be satisfied. □

We now describe the construction of so-called *associated bundles* to a principal bundle, which plays a fundamental role in the description of tubular neighborhoods of orbits of proper actions.

Let $H \to P \to B$ be a principal H-bundle and denote by $\mu_1 \colon P \times H \to P$ the underlying free proper right proper H-action, see Proposition 3.33. Suppose that F is a manifold that admits a left H-action $\mu_2 \colon H \times F \to F$. Then the diagonal action[9]

$$\mu \colon H \times (P \times F) \to (P \times F), \quad \mu\big(h,(x,f)\big) := \big(\mu_1(x,h^{-1}), \mu_2(h,f)\big) \tag{3.14}$$

is a proper left H-action on $P \times F$, whose orbit space we denote by $P \times_H F$. Furthermore, given $(x,f) \in P \times F$, denote its projection to the orbit space by $[x,f] \in P \times_H F$.

Theorem 3.51. *The above orbit space $P \times_H F$ is a manifold, called* twisted space*, and it is the total space of a fiber bundle*

$$F \to P \times_H F \to B,$$

called associated bundle with fiber F*, whose projection $\pi \colon P \times_H F \to B$ is given by $\pi\big([x,f]\big) = \rho(x)$, where $\rho \colon P \to B$ is the projection of the principal H-bundle P, and whose structural group is H.*

Proof (Sketch). Given an H-orbit $H(x)$ in P, let S be a submanifold transverse to $H(x)$ and let U be an H-invariant neighborhood of $H(x)$ in P, as in Claim 3.36.

Claim 3.52. The map $\varphi \colon S \times F \to U \times_H F$, $\varphi(s,f) := [s,f]$, is a diffeomorphism.

Let $\widetilde{\varphi} \colon H \times (S \times F) \to U \times F$ be the restriction of the H-action (3.14) to $S \times F$, that is, $\widetilde{\varphi} := \mu|_{H \times (S \times F)}$. Similarly to Claim 3.36, it is possible to prove that $\widetilde{\varphi}$ is an H-equivariant diffeomorphism, where the H-action on $H \times (S \times F)$ is by left multiplication on the first factor, and the H-action on $U \times F$ is given by (3.14).

$$\begin{array}{ccc} H \times (S \times F) & \xrightarrow{\ \widetilde{\varphi}\ } & U \times F \\ \big\downarrow & & \big\downarrow \\ S \times F & \xrightarrow[\ \varphi\ ]{} & U \times_H F \end{array}$$

Thus, dividing by the H-action, $\widetilde{\varphi}$ descends to the desired diffeomorphism φ. This construction provides charts for the smooth manifold $P \times_H F$.

Let us now define bundle charts for the fiber bundle $F \to P \times_H F \to B$ and verify their compatibility. Let S_1 and S_2 be submanifolds transverse to two different orbits, with $\rho(S_1) \cap \rho(S_2) \neq \emptyset$, and set $\rho_i := \rho|_{S_i} \colon S_i \to \rho(S_i)$. From Claim 3.52,

$$\psi_i \colon \rho(S_i) \times F \to U \times_H F, \quad \psi_i(b,f) := \big[\rho_i^{-1}(b), f\big]$$

[9] Recall Examples 3.5 and 3.8.

are diffeomorphisms. Furthermore, we claim that $(\psi_2^{-1} \circ \psi_1)(b,f) = (b, \theta(b)f)$, where θ is as in the proof of Claim 3.37. In order to verify this, it suffices to show that if $[s_1, f_1] = [s_2, f_2]$, then $f_2 = \mu_2\big(\widetilde{\theta}(s_2), f_1\big)$, where $\widetilde{\theta}$ is also as in the proof of Claim 3.37. The fact that $[s_1, f_1] = [s_2, f_2]$ implies that there exists h such that

$$\mu_2(h, f_1) = f_2, \quad \text{and} \quad \mu_1(s_2, h) = s_1. \tag{3.15}$$

From the proof of Theorem 3.34, $s_1 = \mu_1(s_2, \widetilde{\theta}(s_2))$. Thus, since the action on P is free, it follows that $\widetilde{\theta}(s_2) = h$, which proves the above claim. □

A familiar example of fiber bundle associated to a principal G-bundle is given by the tangent bundle TM of a manifold M. This is the associated bundle with fiber $\mathbb{R}^n$ to the principal G-bundle $B(TM)$, defined in Example 3.29, where $n = \dim M$ and $G = \mathrm{GL}(n,\mathbb{R})$ acts on $\mathbb{R}^n$ by evaluating matrices on vectors.

Exercise 3.53. Define $\psi\colon B(TM) \times_G \mathbb{R}^n \to TM$ by $\psi\big([\xi, v]\big) = \xi(v)$, where $G = \mathrm{GL}(n,\mathbb{R})$ and a frame $\xi \in B(T_xM)$ is interpreted as a linear isomorphism $\xi\colon \mathbb{R}^n \to T_xM$. Prove that ψ is an identification between these two vector bundles.

Hint: To show that ψ is well-defined, note that $\psi\big([\xi \cdot g^{-1}, g(v)]\big) = \xi(v)$ for all $g \in G$. To prove that ψ is injective, verify that $\psi\big([\xi_1, v_1]\big) = \psi\big([\xi_2, v_2]\big)$ if and only if $\xi_2 = \xi_1 \cdot g^{-1}$ and $v_2 = g(v_1)$ for some $g \in G$.

Remark 3.54. More generally, if E is a vector bundle over M of rank n and $B(E)$ is the principal $\mathrm{GL}(n,\mathbb{R})$-bundle of frames of E, then the associated bundle to $B(E)$ with fiber $\mathbb{R}^n$ is canonically identified with the original vector bundle E.

Example 3.55. Let us describe a family of associated bundles with fiber S^2 to the principal S^1-bundle $S^1 \to S^3 \to S^2$, discussed in detail in Exercise 3.31. Writing $S^2 = \{(z,x) \in \mathbb{C} \times \mathbb{R} : |z|^2 + x^2 = 1\}$, define a family μ_k, $k \in \mathbb{N}$, of S^1-actions on S^2,

$$\mu_k\colon S^1 \times S^2 \to S^2, \quad \mu_k\big(e^{i\theta}, (z,x)\big) := \big(e^{ki\theta} z, x\big),$$

that is, μ_k is a rotation of *speed* k on S^2. By Theorem 3.51, the corresponding twisted spaces $M_k = S^3 \times_{S^1} S^2$ are manifolds which are the total space of an S^2-bundle over S^2, that is, $S^2 \to M_k \to S^2$. It is not difficult to show that M_k is diffeomorphic to $S^2 \times S^2$ if k is even, and to $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ if k is odd.[10] In particular, we stress that the notation $P \times_G F$ hides the fact that there may be different G-actions on F which give rise to different twisted spaces.

Exercise 3.56. Reinterpret the isomorphism $(G \times G)/\Delta G \cong G$ described in Example 3.39 in terms of the associated bundle $G \times_G G$ to the trivial principal G-bundle $G \to G \to \{e\}$, where the left G-action on G considered is multiplication on the left.

[10] Recall that $\overline{\mathbb{C}P}^2$ denotes the complex projective plane $\mathbb{C}P^2$ endowed with the orientation opposite to the standard. It is well-known that the connected sum $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ is the *only* nontrivial S^2-bundle over S^2, since such bundles are classified by $\pi_1(\mathrm{Diff}(S^2)) \cong \pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$.

Let μ be a proper G-action on M. Given $x_0 \in M$, let S_{x_0} be a slice at x_0. We define a *tubular neighborhood* of the orbit $G(x_0)$ as the image of S_{x_0} under the G-action, that is,

$$\mathrm{Tub}\big(G(x_0)\big) := \mu\big(G, S_{x_0}\big).$$

The next theorem gives $\mathrm{Tub}\big(G(x_0)\big)$ the structure of an associated fiber bundle; in particular, it shows that manifolds with proper G-actions are *locally* G-equivariant to associated bundles.

Tubular Neighborhood Theorem 3.57. *Let $\mu\colon G \times M \to M$ be a proper action. For every $x_0 \in M$, there exists a G-equivariant diffeomorphism between* $\mathrm{Tub}(G(x_0))$ *and the total space of the associated bundle with fiber S_{x_0},*

$$S_{x_0} \to G \times_H S_{x_0} \to G/H,$$

to the principal H-bundle $H \to G \to G/H$, where $H = G_{x_0}$ is the isotropy at x_0.

Remark 3.58. The G-action considered on $G \times_H S_{x_0}$ is given by $\overline{g} \cdot [g,s] := [\overline{g}g, s]$. Details on the principal H-bundle $H \to G \to G/H$ are found in Corollary 3.38.

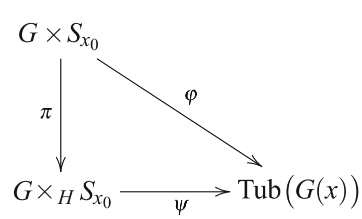
Proof. Let $\varphi\colon G \times S_{x_0} \to \mathrm{Tub}\big(G(x_0)\big)$ be the restriction of the G-action to $G \times S_{x_0}$, that is, $\varphi := \mu|_{G\times S_{x_0}}$. Similarly to the proof of Claim 3.36, note that $\mathrm{d}\varphi_{(e,x)}$ is surjective and $\mathrm{d}\varphi_{(g,x)}(\mathrm{d}L_g X, Y) = \mathrm{d}(\mu^g)_x \circ \mathrm{d}\varphi_{(e,x)}(X,Y)$, hence $\mathrm{d}\varphi_{(g,x)}$ is surjective for all $g \in G$ and $x \in S_{x_0}$. In particular, $\mathrm{Tub}\big(G(x_0)\big)$ is an open neighborhood of $G(x_0)$, which is clearly G-invariant.

Claim 3.59. $\varphi(g,x) = \varphi(h,y)$ if and only if $h = gk^{-1}$ and $y = \mu(k,x)$, where $k \in H$.

Assume that $\varphi(g,x) = \varphi(h,y)$. Then $\mu(g,x) = \mu(h,y)$ and $y = \mu(k,x)$, where $k = h^{-1}g$. Since $x, y \in S_{x_0}$, it follows from Definition 3.47 that $k \in H$. The converse statement is clear. We now define the candidate to G-equivariant diffeomorphism,

$$\psi\colon G \times_H S_{x_0} \to \mathrm{Tub}\big(G(x_0)\big), \quad \psi\big([g,s]\big) := \mu(g,s). \tag{3.16}$$

The fact that $\pi\colon G \times S_{x_0} \to G \times_H S_{x_0}$, $\pi(g,s) = [g,s]$, is the projection of a principal bundle, and Claim 3.59, imply that ψ is well-defined, smooth, bijective and G-equivariant. It only remains to prove that ψ is a diffeomorphism.



Note that $\mathrm{d}\pi$ and $\mathrm{d}\varphi$ are surjective, and $\varphi = \psi \circ \pi$, hence $\mathrm{d}\psi$ is surjective. As

$$\dim G \times_H S_{x_0} = \dim G/H + \dim S_{x_0} = \dim M = \dim \mathrm{Tub}(G(x_0)), \tag{3.17}$$

it follows that $\mathrm{d}\psi$ is an isomorphism. Thus, by the Inverse Function Theorem, ψ is a local diffeomorphism, and hence a diffeomorphism since it is bijective. □

Remark 3.60. By the above result, the image of a slice at x_0 under the transformation $\mu^g\colon M \to M$ is a slice at $\mu(g,x_0)$, that is, $S_{\mu(g,x_0)} = \mu^g(S_{x_0})$. It also follows from the Tubular Neighborhood Theorem 3.57 that there is a unique G-equivariant retraction $r\colon \mathrm{Tub}\big(G(x_0)\big) \to G(x_0)$ such that $r \circ \psi = \widetilde{\mu}_{x_0} \circ \pi$, where $\pi\colon G \times_H S_{x_0} \to G/H$ is the projection map of the associated bundle, $\widetilde{\mu}_{x_0}$ is defined as in Proposition 3.41, and ψ is the G-equivariant diffeomorphism (3.16).

Remark 3.61. Let $\mu\colon G \times M \to M$ be an action whose orbits have constant dimension. The Tubular Neighborhood Theorem 3.57 implies that the holonomy group $\mathrm{Hol}(G(x_0),x_0)$ of the leaf $G(x_0)$ of the foliation $\{G(x)\}_{x\in M}$ coincides with the image of the *slice representation* of $H = G_{x_0}$, see Definition 3.72 and Remarks 5.13 and 5.14. In the general case of foliations with compact leaves and finite holonomy, there is an analogous result to the Tubular Neighborhood Theorem 3.57, known as the *Reeb Local Stability Theorem* (see Moerdijk and Mrčun [163]).

3.3 Isometric Actions

In this section, we study the relation between *proper* and *isometric* actions. In Proposition 3.62, we show that actions by closed subgroups of isometries are proper, and conversely, in Theorem 3.65, we show that every proper action can be made isometric with respect to a certain Riemannian metric (see also Remark 3.67).

Proposition 3.62. *Let (M,g) be a Riemannian manifold and G a closed subgroup of the isometry group* $\mathrm{Iso}(M,\mathrm{g})$*. The action $\mu\colon G\times M \ni (g,x) \mapsto g(x) \in M$ is proper.*

Proof. For the sake of brevity, we only prove the result for the case in which (M,g) is complete. Consider sequences $\{g_n\}$ in G and $\{x_n\}$ in M, such that $\lim \mu(g_n,x_n) = y$ and $\lim x_n = x$. We have to prove that there exists a convergent subsequence of $\{g_n\}$ in G. Choose n_0 such that $\mathrm{dist}\big(y,\mu(g_n,x_n)\big) < \frac{\varepsilon}{2}$ and $\mathrm{dist}(x,x_n) < \frac{\varepsilon}{2}$ for $n > n_0$.

Claim 3.63. For fixed $R_1 > 0$, we have $\mu\big(g_n, B_{R_1}(x)\big) \subset B_{R_1+\varepsilon}(y)$ for all $n > n_0$.

Indeed, for $z \in B_{R_1}(x)$, we have

$$\begin{aligned}
\mathrm{dist}\big(\mu(g_n,z),y\big) &\le \mathrm{dist}\big(y,\mu(g_n,x_n)\big) + \mathrm{dist}\big(\mu(g_n,x_n),\mu(g_n,z)\big)\\
&< \tfrac{\varepsilon}{2} + \mathrm{dist}(x_n,z)\\
&\le \tfrac{\varepsilon}{2} + \mathrm{dist}(z,x) + \mathrm{dist}(x,x_n)\\
&< \varepsilon + R_1.
\end{aligned}$$

The fact that each g_n is an isometry of (M,g) implies that $\{g_n\}$ is an equicontinuous family. Thus, by the Arzelà-Ascoli Theorem,[11] Claim 3.63, and the fact that closed balls in M are compact (since M was assumed complete), there exists a subsequence $\{g_n^{(1)}\}$ of $\{g_n\}$ that converges uniformly on $\overline{B_{R_1}(x)}$ to a continuous map from $\overline{B_{R_1}(x)}$ to M. Using this argument inductively, define a subsequence $\{g_n^{(i)}\}$ of $\{g_n^{(i-1)}\}$ that converges uniformly on $\overline{B_{R_i}(x)}$ to a continuous map from $\overline{B_{R_i}(x)}$ to M, where $R_i > R_{i-1}$. Note that the *diagonal* subsequence $\{g_i^{(i)}\}$ converges uniformly on each $\overline{B_{R_i}(x)}$ to a continuous map $g\colon M \to M$. It follows from the Myers-Steenrod Theorem 2.12 that g is an isometry of (M,g) that belongs to G. $\square$

Remark 3.64. The hypothesis that G is a *closed* subgroup of $\mathrm{Iso}(M,\mathrm{g})$ is crucial for Proposition 3.62 to hold. For instance, the $\mathbb{R}$-action μ_2 on the Euclidean space $\mathbb{R}^2$ defined in Example 3.43 is clearly isometric, and hence so is the induced $\mathbb{R}$-action $\widetilde{\mu}_2$ on the flat torus T^2. However, $\widetilde{\mu}_2$ is not a proper action, see Exercise 3.18.

Using the Slice Theorem 3.49, we now prove a converse result, following Palais and Terng [182, Chapter 5].

Theorem 3.65. *Let $\mu\colon G\times M \to M$ be a proper action. There exists a G-invariant metric* g *on M such that $\mu^G = \{\mu^g : g\in G\}$ is a closed subgroup of* $\mathrm{Iso}(M,\mathrm{g})$.

Proof. The strategy to construct g is to define it locally, using an averaging procedure similar to (3.12), and then glue these together using G-invariant partitions of unity. Note that M/G is paracompact,[12] since it is a Hausdorff locally compact space and the union of countably many compact spaces. Thus, there exists a locally finite open covering $\{\pi(S_{x_\alpha})\}$ of M/G, where S_{x_α} is a slice at a point $x_\alpha \in M$. Set $U_\alpha := \pi^{-1}(\pi(S_{x_\alpha}))$ and let $\{f_\alpha\}$ be a G-invariant partition of unity subordinate to $\{U_\alpha\}$, which exists by the following result of Palais [180].

Claim 3.66. If $\{U_\alpha\}$ is a locally finite open cover of M by G-invariant open sets, then there exists a smooth partition of unity $\{f_\alpha\}$ subordinate to $\{U_\alpha\}$, such that each $f_\alpha\colon U_\alpha \to [0,1]$ is G-invariant.

Define a Riemannian metric $\langle\cdot,\cdot\rangle^\alpha$ on M along the slice S_{x_α}, i.e., a section of $(TM^*\otimes TM^*)|_{S_{x_\alpha}}$, by setting

$$\langle X,Y\rangle_p^\alpha := \int_{H_\alpha} \langle\langle \mathrm{d}\mu^g X, \mathrm{d}\mu^g Y\rangle\rangle_{\mu(g,p)}\, \omega,$$

[11] This theorem gives a criterion for convergence of continuous maps in the compact-open topology. More precisely, a sequence of continuous functions $\{g_n\colon K\to B\}$ between compact metric spaces K and B admits a uniformly convergent subsequence if $\{g_n\}$ is *equicontinuous*, i.e., for every $\varepsilon>0$ there exists $\delta>0$ such that $\mathrm{dist}(g_n(x),g_n(y))<\varepsilon$ for all $\mathrm{dist}(x,y)<\delta$, $x,y\in K$, $n\in\mathbb{N}$.

[12] This means that any open cover of M/G admits a locally finite refinement.

where ω is a right-invariant volume form on the isotropy group $H_\alpha = G_{x_\alpha}$ and $\langle\langle\cdot,\cdot\rangle\rangle$ is an arbitrary Riemannian metric on S_{x_α}. It is easy to see that $\langle\cdot,\cdot\rangle^\alpha$ is H_α-invariant. Define a G-invariant Riemannian metric g_α on U_α by setting

$$\mathrm{g}_\alpha\big(\mathrm{d}\mu^g X, \mathrm{d}\mu^g Y\big)_{\mu(g,p)} := \langle X,Y\rangle_p^\alpha. \tag{3.18}$$

The above metric is well-defined, since $\langle\cdot,\cdot\rangle^\alpha$ is H_α-invariant on S_{x_α}. Finally, define the desired metric g on M as $\mathrm{g} := \sum_\alpha f_\alpha \mathrm{g}_\alpha$.

It remains only to verify that μ^G is a closed subgroup of $\mathrm{Iso}(M,\mathrm{g})$. Assume that μ^{g_n} converges uniformly in each compact of M to an isometry $f\colon M \to M$. Thus, for each $x \in M$ we have that $\lim \mu(g_n, x) = f(x)$. Properness of the action implies that a subsequence $\{g_{n_i}\}$ converges to $g \in G$. It is not difficult to check that if $\lim g_{n_i} = g$, then $\mu^{g_{n_i}}$ converges uniformly in each compact of M to μ^g, and hence $\mu^g = f$. □

Remark 3.67. Theorem 3.65 clearly provides a type of converse statement to Proposition 3.62. Nevertheless, there are two subtle issues that might prevent one from exchanging isometric actions by proper actions when proving results about these. First, and most important, the G-invariant metric g provided by Theorem 3.65 is not necessarily complete.[13] We stress that completeness is a necessary hypothesis in many results for isometric actions. Second, the closed subgroup μ^G of $\mathrm{Iso}(M,\mathrm{g})$ is only isomorphic to G if $\mu\colon G\times M \to M$ is effective, see Remark 3.3.

Remark 3.68. Theorem 3.65 can be generalized to the context of *proper groupoids*, see Pflaum, Posthuma, and Tang [185]. These objects admit a Riemannian metric such that the induced foliation is a singular Riemannian foliation. Similarly to Remark 3.67, the ambient space with this invariant metric may fail to be complete, and many results on Riemannian foliations that assume completeness do not hold. For example, orbits of proper groupoids with trivial holonomy may not be diffeomorphic to each other. Nevertheless, when M is compact, foliations induced by a proper groupoid seem to be a particular case of *orbit-like foliations* on compact manifolds, i.e., singular Riemannian foliations whose restriction to slices are homogenous foliations, see [16] and Chap. 5.

Exercise 3.69 ($\star$). Let $\mu\colon G\times M\to M$ be an effective isometric proper action. Prove that if $\dim M = n$, then $\dim G \leq \frac{n(n+1)}{2}$.

Hint: Use that $\dim G - \dim G_x = \dim G(x) \leq \dim M = n$.

Given an isometric G-action on a Riemannian manifold (M,g), we usually refer to the directions tangent to the orbits as *vertical directions* and directions normal to the orbits as *horizontal directions*. In this way, we define the *vertical space* at $x \in M$ as

$$\mathscr{V}_x := T_x G(x) = \{X^*(x) : X \in \mathfrak{g}\}, \tag{3.19}$$

[13] Note that completeness can be guaranteed if, e.g., M is compact.

see (3.11), and the *horizontal space* as

$$\mathscr{H}_x := \{v \in T_xM : \mathrm{g}(v, X^*(x)) = 0 \text{ for all } X \in \mathfrak{g}\}, \tag{3.20}$$

cf. (2.10) and (2.11). In this way, we get an orthogonal splitting $T_xM = \mathscr{V}_x \oplus \mathscr{H}_x$. Note that the dimensions of $\mathscr{V}_x$ and $\mathscr{H}_x$ may vary with $x \in M$, so these are *not* distributions on M. However, they become distributions when restricted to each *stratum* of M, see Theorem 3.102. Accordingly, we say that a curve $\gamma\colon [a,b] \to M$ is *vertical* or *horizontal* if its tangent vector γ' is always vertical or horizontal, respectively.

The following result for isometric actions was observed by Kleiner in his PhD thesis [141], and is usually referred to as *Kleiner's lemma*.

Lemma 3.70. *Let $\mu: G \times M \to M$ be an isometric action and let $\gamma\colon [0,1] \to M$ be a geodesic segment between $G(\gamma(0))$ and $G(\gamma(1))$ that realizes the distance*[14] *between these orbits. There exists a subgroup H of G such that $G_{\gamma(t)} = H$ for $t \in (0,1)$ and H is a subgroup of $G_{\gamma(0)}$ and $G_{\gamma(1)}$.*

Proof. Let $H := \big\{g \in G : \mu(g, \gamma(t)) = \gamma(t) \text{ for all } t \in [0,1]\big\}$. Suppose there exists some $t_0 \in (0,1)$ and $g \in G_{\gamma(t_0)}$ such that $g \notin H$. In particular, $\mathrm{d}(\mu^g)_{\gamma(t_0)}(\dot\gamma(t_0)) \neq \dot\gamma(t_0)$. Define a piecewise smooth path $\widetilde{\gamma}\colon [0,1] \to M$ by setting $\widetilde{\gamma}|_{[0,t_0]} := \gamma_{[0,t_0]}$ and $\widetilde{\gamma}|_{[t_0,1]} := \mu^g\big(\gamma_{[t_0,1]}\big)$. Then $\widetilde{\gamma}$ joins $G(\gamma(0))$ to $G(\gamma(1))$ and has the same length as γ, contradicting the fact that minimizing geodesic segments are smooth. □

Exercise 3.71. Use the isometric action in $\mathbb{R}^3$ described in Example 3.4 (iii) to show that the conclusion of Lemma 3.70 fails if γ is only a minimal geodesic segment.

We conclude this section with the important definition of *slice representation*, which is the restriction of the isotropy representation of an isometric action to the normal space to the orbit, see Exercise 3.7. More precisely, if $\mu: G \times M \to M$ is a proper isometric action, note that $S_x = \exp_x\big(B_\varepsilon(0)\big)$ is a slice at x, called *normal slice*, where $B_\varepsilon(0) \subset \nu_x G(x)$ is an open ball in the normal space to $G(x)$, cf. (3.13).

Definition 3.72. Let $\mu: G \times M \to M$ be an isometric proper action and S_x a normal slice at x. The *slice representation* of G_x is the linear (orthogonal) representation

$$G_x \ni g \longmapsto \mathrm{d}(\mu^g)_x \in \mathrm{O}\big(\nu_x G(x)\big) \subset \mathrm{GL}\big(\nu_x G(x)\big).$$

[14] Note that the assumption that γ realizes the distance between the orbits $G(\gamma(0))$ and $G(\gamma(1))$ is stronger than γ being a minimal geodesic segment between its endpoints $\gamma(0)$ and $\gamma(1)$.

3.4 Principal Orbits

In this section, we study some geometric properties of principal orbits. In particular, we prove the Principal Orbit Theorem 3.82, that implies that the set of points in principal orbits is open and dense.

Definition 3.73. Let $\mu : G \times M \to M$ be a proper action. Then $G(x)$ is a *principal orbit* if there exists a neighborhood V of x in M such that for each $y \in V$, $G_x \subset G_{\mu(g,y)}$ for some $g \in G$.

In rough terms, principal orbits are the ones that have the *smallest* isotropy group (i.e., the largest *type*, see Definition 3.84) among the nearby orbits.

Proposition 3.74. *Let $\mu : G \times M \to M$ be a proper left action. Then the following are equivalent:*

(i) *$G(x)$ is a principal orbit;*
(ii) *If S_x is a slice at x, then $G_x = G_y$ for all $y \in S_x$.*

Proof. First, let us prove that (i) implies (ii). From the definition of slice, for each $y \in S_x$ we have $G_y \subset G_x$. On the other hand, since $G(x)$ is a principal orbit, there exists g such that $g G_x g^{-1} \subset G_y$. Thus, $g G_x g^{-1} \subset G_y \subset G_x$. Since G_x is compact, we conclude that the above inclusions are equalities. In particular, $G_y = G_x$.

To prove the converse, we have to prove that for z in a tubular neighborhood of $G(x)$, there exists g such that $G_x \subset g G_z g^{-1}$. The orbit $G(z)$ intersects the slice S_x at least at one point, say y. Since y and z are in the same orbit, there exists g such that $G_y = g G_z g^{-1}$, recall Exercise 3.9. Therefore $G_x = G_y = g G_z g^{-1}$. □

Remark 3.75. As we will see in Chap. 5, the above proposition implies that each principal orbit is a leaf with trivial holonomy of the foliation $\mathscr{F} = \{G(x)\}_{x \in M}$.

Exercise 3.76. Let $\mu : G \times M \to M$ be a proper action, and let $K := \bigcap_{x \in M} G_x$ be the ineffective kernel of the action. Consider the induced effective action $\widetilde{\mu} : G/K \times M \to M$, see Remark 3.3. Prove that $\widetilde{\mu}$ is smooth and proper. Check that if an orbit is principal with respect to $\widetilde{\mu}$, then it is also principal with respect to μ.

Exercise 3.77. Let $\mu : G \times M \to M$ be an isometric proper action and S_x a normal slice at x. Prove that $G(x)$ is a principal orbit if and only if the slice representation of G_x is trivial.

Let us explore some geometric properties of orbits of isometric actions.

Proposition 3.78. *Let $\mu : G \times M \to M$ be an isometric proper action on a complete Riemannian manifold M. Then the following hold:*

(i) *A geodesic γ orthogonal to an orbit $G(\gamma(0))$ remains orthogonal to all orbits it intersects, i.e., γ is a horizontal geodesic.*

In the following, suppose $G(x)$ is a principal orbit:

(ii) *Given $\xi \in \nu_x G(x)$, $\widehat{\xi}_{\mu(g,x)} := \mathrm{d}(\mu^g)_x \xi$ is a well-defined normal vector field along $G(x)$, called* equivariant normal field*;*

(iii) $\mathscr{S}_{\widehat{\xi}_{\mu(g,x)}} = \mathrm{d}\mu^g \mathscr{S}_{\widehat{\xi}_x} \mathrm{d}\mu^{g^{-1}}$*, where $\mathscr{S}_{\widehat{\xi}}$ is the shape operator of $G(x)$;*

(iv) *Principal curvatures of $G(x)$ with respect to an equivariant normal field $\widehat{\xi}$ are constant along $G(x)$;*

(v) $\big\{\exp_y\big(\widehat{\xi}_y\big) : y \in G(x)\big\}$ *is an orbit of μ.*

Proof. In order to prove item (i), by Proposition 3.13 and (3.11), it suffices to prove that if a Killing vector field X is orthogonal to $\gamma'(0)$, then X is orthogonal to $\gamma'(t)$ for all t. Since $\langle \nabla_{\gamma'(t)} X, \gamma'(t)\rangle = 0$ (see Proposition 2.14) and γ is a geodesic, we have that $\frac{\mathrm{d}}{\mathrm{d}t}\langle X, \gamma'(t)\rangle = 0$. Thus, X is always orthogonal to $\gamma'(t)$.

Item (ii) follows from Exercise 3.77. Item (iii) follows from:

$$\begin{aligned}
\big\langle \mathrm{d}\mu^{g^{-1}} \mathscr{S}_{\widehat{\xi}_{\mu(g,x)}} \mathrm{d}\mu^g W, Z\big\rangle_x &= \big\langle \mathscr{S}_{\widehat{\xi}_{\mu(g,x)}} \mathrm{d}\mu^g W, \mathrm{d}\mu^g Z\big\rangle_{\mu(g,x)} \\
&= \big\langle -\nabla_{\mathrm{d}\mu^g W} \mathrm{d}(\mu^g)_x \xi, \mathrm{d}\mu^g Z\big\rangle_{\mu(g,x)} \\
&= \big\langle -\nabla_W \widehat{\xi}, Z\big\rangle_x \\
&= \big\langle \mathscr{S}_{\widehat{\xi}_x} W, Z\big\rangle_x.
\end{aligned}$$

As for (iv), note that if $\mathscr{S}_{\widehat{\xi}} X = \lambda X$, then $\mathrm{d}\mu^{g^{-1}} \mathscr{S}_{\widehat{\xi}_{\mu(g,x)}} \mathrm{d}\mu^g X = \lambda X$. Hence $\mathscr{S}_{\widehat{\xi}_{\mu(g,x)}} \mathrm{d}\mu^g X = \lambda \mathrm{d}\mu^g X$. Finally, item (v) follows from (2.2), because $\exp_{\mu(g,x)}\big(\widehat{\xi}_{\mu(g,x)}\big) = \exp_{\mu(g,x)}(\mathrm{d}\mu^g \xi_x) = \mu^g(\exp_x(\xi))$. □

Remark 3.79. The above proposition illustrates a few concepts and results of Chaps. 4 and 5. Item (i) implies that the homogeneous foliation (3.4) by orbits of a proper isometric action is a *singular Riemannian foliation* (see Definition 5.2). Item (v) implies that one can reconstruct this foliation taking all parallel submanifolds to a principal orbit. This is a consequence of *equifocality*, which is valid for every singular Riemannian foliation. Item (iv) and the fact that equivariant normal fields are parallel normal fields when the action is *polar* imply that principal orbits of a polar action on Euclidean space are *isoparametric* (see Definitions 4.8 and 4.14).

Let us explore a little further item (i) of Proposition 3.78 to obtain some metric information about the orbit space M/G of an isometric proper action. If a geodesic segment $\gamma\colon [0,1] \to M$ minimizes distance between the orbit $G(\gamma(0))$ and $\gamma(1)$, then it must be orthogonal to the submanifold $G(\gamma(0))$. Hence, by the above, γ must be horizontal. Now, for any $g \in G$, since $\mu^g\colon M \to M$ is an isometry, the curve $(\mu^g \circ \gamma)\colon [0,1] \to M$ is a horizontal minimal geodesic segment joining $G(\gamma(0))$ and $\mu(g,\gamma(1)) \in G(\gamma(1))$, with the same length as γ. This means that for any $g \in G$, the distances from $G(\gamma(0))$ to $\gamma(1)$ and $\mu(g,\gamma(1))$ are the same. Thus, orbits of an isometric group action are *equidistant*, and hence there is a well-defined *orbital distance*, i.e., a distance on M/G. The orbit space M/G equipped with this distance becomes a metric space, more precisely, a *length metric space*.[15]

Remark 3.80. There are many important aspects of the geometry of the metric space M/G that reflect the geometry of an isometric G-action on M, especially if M has a lower curvature bound, making M/G an *Alexandrov space*.[16] Much recent progress on areas involving the interface of transformation groups and Riemannian geometry was achieved through the use of Alexandrov geometry of orbit spaces; however, we do not go in this direction in this book. For a brief glance at some of these ideas, see Example 3.107, where the appropriate generalization of tangent space at a nonregular point in M/G, called *tangent cone*, is mentioned in a specific example. Details on how Alexandrov techniques apply to study isometric actions can be found in the survey of Grove [107].

Exercise 3.81. Let $\mu\colon G\times M\to M$ be a free isometric action and recall Theorem 3.34. Show that the quotient map $\pi\colon M\to M/G$ is Riemannian submersion, where M/G is equipped with the natural orbit metric, whose associated distance is the above orbital distance.[17]

We conclude this section proving the so-called Principal Orbit Theorem, which guarantees that the subset of points in M on principal orbits is open and dense, and that M/G has an open and dense subset which is a smooth manifold (see Theorem 3.95 and Remark 3.106).

Principal Orbit Theorem 3.82. *Let $\mu\colon G\times M\to M$ be a proper action, where M is connected, and denote by M_{princ} the set of points of M contained in principal orbits. Then the following hold:*

(i) *M_{princ} is open and dense in M;*
(ii) *The subset M_{princ}/G of M/G is a connected manifold;*
(iii) *If $G(x)$ and $G(y)$ are principal orbits, there exists $g\in G$ such that $G_x=gG_yg^{-1}$.*

Proof. We first prove the existence of a principal orbit. Since G has finite dimension and the isotropy groups are compact, we can choose $x\in M$ such that G_x has the lowest dimension among isotropy groups and, for that dimension, the smallest number of connected components. If S_x is a slice at x, by definition, $G_y\subset G_x$ for every $y\in S_x$. By construction, we conclude that $G_y=G_x$. Hence, from Proposition 3.74, $G(x)$ is a principal orbit.

In order to prove that M_{princ} is open, let $x\in M_{\text{princ}}$ and let S_x be a slice at x. Proposition 3.74 implies that $G_y=G_x$ for every $y\in S_x$. We claim that each $y\in S_x$ belongs to a principal orbit. If a point z is close to y, then z is in a tubular

[15] A metric space is a length metric space if the distance between any two points is realized by a shortest curve, called a geodesic.

[16] An *Alexandrov space* is a finite-dimensional length metric space with a lower curvature bound, in a comparison geometry sense. For details, see [59, 107].

[17] More generally, if μ is not free, then $\pi\colon M\to M/G$ is a *submetry*, which is a generalization of submersions to metric spaces.

neighborhood of $G(x)$. Note that $G(z)$ intersects S_x in at least one point, so there exists g such that $\mu(g,z) = w \in S_x$. Thus, $G_y = G_x = G_w = g\,G_z\,g^{-1}$ and hence each $y \in S_x$ belongs to a principal orbit, as claimed. Therefore, every point in the tubular neighborhood $\mathrm{Tub}(G(x)) = \mu(G, S_x)$ belongs to a principal orbit, proving M_{princ} is open.

If μ is an *isometric* action on a *complete* Riemannian manifold, then the rest of the proof follows easily from Proposition 3.74 and Kleiner's Lemma 3.70, since given any two orbits, there exists a geodesic segment that realizes the distance between these orbits. In this case, one can also prove that the space M_{princ}/G is convex. We encourage the reader to verify these claims, and, in what follows, we finish the proof in the general case.

To prove that M_{princ} is dense, consider $p \notin M_{\mathrm{princ}}$ and U a neighborhood of p. Choose $x \in U$ such that G_x has the lowest dimension among isotropy groups and, for that dimension, the smallest number of connected components. Then, from the argument above, we conclude that $x \in M_{\mathrm{princ}}$.

Proposition 3.74 and some arguments from the proof of Theorem 3.34 can be used to prove that M_{princ}/G is a manifold. In order to prove that M_{princ}/G is connected, assume that the action is proper and isometric. A set $A \subset M/G$ *does not locally disconnect* M/G if each $p \in M/G$ has a neighborhood U such that $U \setminus A$ is path-connected. Using (i), it is easy to verify that if $(M \setminus M_{\mathrm{princ}})/G$ does not locally disconnect M/G, then M_{princ}/G is path-connected. Thus, it suffices to prove that $(M \setminus M_{\mathrm{princ}})/G$ does not locally disconnect M/G. This can be done using the fact that $S_x/G_x = \mathrm{Tub}(G(x))/G$, the slice representation, Exercise 3.76 and the next:

Claim 3.83. Let K be a closed subgroup of $\mathrm{O}(n)$, acting on $\mathbb{R}^n$ by multiplication. Then $\mathbb{R}^n_{\mathrm{princ}}/K$ is path-connected.

We prove Claim 3.83 by induction. If $n = 1$, then $K = \mathbb{Z}_2$ or $K = \{1\}$. In both cases, $\mathbb{R}_{\mathrm{princ}}/K$ is path-connected. For each sphere S^{n-1} centered at the origin, we can apply the induction hypothesis and the slice representation to conclude that $(S^{n-1} \setminus S^{n-1}_{\mathrm{princ}})/K$ does not locally disconnect S^{n-1}/K. Therefore, $S^{n-1}_{\mathrm{princ}}/K$ is path-connected. Consider $x, y \in \mathbb{R}^n_{\mathrm{princ}}$ and let $\widetilde{x}$ be the projection of x on the sphere S^{n-1} that contains y. The points along the straight line segment that joins x to $\widetilde{x}$ belong to principal orbits. Thus, $K(x)$ and $K(\widetilde{x})$ are connected by a path in $\mathbb{R}^n_{\mathrm{princ}}/K$. As we have already proved, $K(y)$ and $K(\widetilde{x})$ are connected by a path in $S^{n-1}_{\mathrm{princ}}/K$ and hence in $\mathbb{R}^n_{\mathrm{princ}}/K$. Therefore $K(x)$ and $K(y)$ are connected by a path in $\mathbb{R}^n_{\mathrm{princ}}/K$.

Finally, item (iii) is a direct consequence of item (ii). □

3.5 Orbit Types

The *principal orbits* discussed in the last section are one of many possible *orbit types*. We now discuss some properties of orbit types of proper actions, e.g., we prove that on compact manifolds there is only a finite number of these, see

Theorem 3.91. In addition, it is also proved that each connected component of a set of orbits of the same type is a connected component of the total space of a certain fiber bundle, see Theorem 3.95. Finally, it is also proved that the connected components of sets of orbits of the same type give a *stratification* of M, see Theorem 3.102.

Definition 3.84. Let $\mu\colon G\times M\to M$ be a proper action.

(i) The orbit $G(x)$ has a *larger orbit type* than $G(y)$ if there is a G-equivariant map $\varphi\colon G(x)\to G(y)$, or, equivalently, if there is $g\in G$ such that $G_x\subset G_{\mu(g,y)}$;
(ii) The orbits $G(x)$ and $G(y)$ have the *same orbit type* if there is a G-equivariant diffeomorphism $\varphi\colon G(x)\to G(y)$, or, equivalently, if there is $g\in G$ such that $G_x=G_{\mu(g,y)}$;
(iii) An orbit $G(x)$ is said to be *regular* if the dimension of $G(x)$ coincides with the dimension of principal orbits;
(iv) A nonprincipal regular orbit is called *exceptional*;
(v) A nonregular orbit is called *singular*.[18]

These definitions are also used referring to points, e.g., x and y have the *same orbit type* if $G(x)$ and $G(y)$ have the same type, and x is *singular* if $G(x)$ is singular.

It is not difficult to see that $G(x)$ and $G(y)$ have the same orbit type if and only if $G(x)$ has a larger orbit type than $G(y)$ and $G(y)$ has a larger orbit type than $G(x)$, see Exercise 3.9. It is also clear that having the same orbit type is an equivalence relation (on both M/G and M). Note that Theorem 3.82 asserts not only that M_{princ} is open and dense in M, but also that there exists a unique type of principal orbit.

Exercise 3.85. Let $\mu\colon G\times M\to M$ be a proper action, and let $K:=\bigcap_{x\in M}G_x$ be the ineffective kernel of this action. Consider the induced effective action $\widetilde{\mu}\colon G/K\times M\to M$, see Remark 3.3 and Exercise 3.76. Prove that orbits of the same type with respect to $\widetilde{\mu}$ are also of the same type with respect to μ.

Exercise 3.86. Let $\mu\colon G\times M\to M$ be a proper action and $G(p)$ a principal orbit. Prove that $G(x)$ is an exceptional orbit if and only if $\dim G(x)=\dim G(p)$ and the number of connected components of G_x is greater than the number of connected components of G_p.

Exercise 3.87. Consider the isometric action $\widetilde{\mu}\colon S^1\times S^2\to S^2$ of the circle S^1 on the round sphere S^2 by rotations. Use Exercise 3.40 to verify that $\widetilde{\mu}$ induces an isometric action $\mu\colon S^1\times\mathbb{R}P^2\to\mathbb{R}P^2$, which has an exceptional orbit.

Exercise 3.88 ($\star$). Consider the action of $\mathrm{SU}(3)$ on itself by conjugation. Prove that the orbits of this action are diffeomorphic to one of the following:

[18] It is important to notice that *singular* orbits are smooth submanifolds, whose *singular* nature is simply to have a lower dimension.

(i) $\{\lambda I\}$, where $I \in \mathrm{SU}(3)$ is the identity and $\lambda \in \mathbb{C}$ satisfies $\lambda^3 = 1$;
(ii) $\mathrm{SU}(3)/S(\mathrm{U}(2) \times \mathrm{U}(1))$;
(iii) $\mathrm{SU}(3)/T$, where T is the subgroup of diagonal matrices in $\mathrm{SU}(3)$.

Conclude, using Exercise 3.44 and Remark 3.46, that an orbit is diffeomorphic to either a point, or complex projective plane $\mathbb{C}P^2$, or the complex flag manifold $\mathrm{SU}(3)/T$.

Hint: Use the fact that each matrix of $\mathrm{SU}(3)$ is conjugate to a matrix of T, by diagonalization.

Remark 3.89. As explained in the next chapter, Exercises 3.87 and 3.88 give examples of polar actions. An isometric action is a *polar action* if for each regular point x, the set $\exp_x(\nu_x G(x))$ is a (totally geodesic) submanifold that intersects every orbit orthogonally, see Definition 4.8 for details. In Exercise 3.87, the polar action admits an exceptional orbit, and in Exercise 3.88, the polar action admits only principal and singular orbits. Note that $\mathbb{R}P^2$ is not simply-connected and $\mathrm{SU}(3)$ is simply-connected. As explained in Chap. 5, polar actions do not admit exceptional orbits if the ambient space is simply-connected (Corollary 5.35).

Remark 3.90. For an example of nonpolar isometric action, see Example 3.11.

Theorem 3.91. *Let $\mu : G \times M \to M$ be a proper action. For each $x \in M$, there exists a slice S_x such that the tubular neighborhood $\mathrm{Tub}(G(x)) = \mu(G, S_x)$ contains only finitely many different orbit types. In particular, if M is compact, there is only a finite number of different orbit types in M.*

Proof. Using Theorem 3.65, let g be a Riemannian metric on M such that $\mu^G \subset \mathrm{Iso}(M, \mathrm{g})$. It is not difficult to verify the following:

Claim 3.92. If $y, z \in S_x$ have the same G_x-orbit type, then y and z have the same G-orbit type.

Therefore, it suffices to prove the result for the G_x-action on S_x. By the slice representation (see Definition 3.72) and Exercise 3.85, we can further reduce the problem to proving the result for a linear (orthogonal) action of $K \subset \mathrm{O}(n)$ on $\mathbb{R}^n$ by multiplication. With this goal, we proceed by induction on n. If $n = 1$, then $K = \mathbb{Z}_2$ or $K = \{1\}$. In both cases, there exists only a finite number of different orbit types (at most 2). For a sphere S^{n-1}, we can apply the induction hypothesis, Claim 3.92 and the slice representation, to conclude that there exists a finite number of K-orbit types. For each $p \in S^{n-1}$, the points along the straight line segment joining the origin to p (except for the origin itself) have the same isotropy type (cf. Kleiner's Lemma 3.70). Thus, there exists a finite number of K-orbit types in $\mathbb{R}^n$. □

In order to study how different orbit types of a proper action *stratify* a manifold, we first need to develop structural results for sets of points that have the same orbit type. The first step in this direction is the following result on fixed point sets.

Proposition 3.93. *Let $\mu \colon G \times M \to M$ be a proper action and $H \subset G$ a subgroup. The connected components of the* fixed point set

$$M^H := \big\{x \in M : \mu(h,x) = x, \text{ for all } h \in H\big\}$$

are embedded submanifolds of M (of possibly different dimensions), whose tangent space is given by

$$T_x M^H = \big\{v \in T_x M : \mathrm{d}(\mu^h)_p v = v, \text{ for all } h \in H\big\}.$$

In addition, if the action μ is isometric, then these submanifolds are totally geodesic.

Proof. The fixed point set of H coincides with the fixed point set of its closure $\overline{H}$, and hence we may assume H is a closed subgroup of G. From Theorem 3.65, there exists an H-invariant Riemannian metric g on M. Given $x \in M^H$, let $\varepsilon > 0$ be such that $\exp_x \colon T_x M \to M$ is a diffeomorphism from the open ball $B_\varepsilon(0) \subset T_x M$ onto its image. Notice that, from (2.2), this diffeomorphism is H-equivariant, where the H-action on $B_\varepsilon(0)$ is the isotropy representation defined in Exercise 3.7. In particular, the exponential map provides a submanifold chart for M^H near x, which is mapped to the linear subspace $T_x M^H$ of $T_x M$. This proves that each connected component of M^H is an embedded submanifold. Finally, it is easy to deduce from (2.2) that each of these submanifolds is totally geodesic if the G-action is isometric. □

Besides the notion of orbits (or points) that have the *same orbit type*, it is convenient to define the following refinement, containing more geometric information.[19]

Definition 3.94. Two orbits $G(x)$ and $G(y)$ of a proper action have the *same local orbit type* if there is a G-equivariant diffeomorphism $\varphi \colon \mathrm{Tub}\big(G(x)\big) \to \mathrm{Tub}\big(G(y)\big)$.

Thus, we have 2 equivalence relations: *to have the same orbit type* and *to have the same local orbit type*, denoted respectively by $\sim$ and $\approx$, following [79]. Clearly, having the same local orbit type is a stronger condition than having the same orbit type (cf. Definition 3.84), so equivalence classes with respect to $\sim$ are partitioned into equivalence classes with respect to $\approx$. These equivalence classes are denoted

$$M_x^{\sim} := \{y \in M : x \sim y\} \quad \text{and} \quad M_x^{\approx} := \{y \in M : x \approx y\},$$

and respectively called *orbit type* and *local orbit type* of $x \in M$. In this way, an orbit type is the union of local orbit types. Using the Tubular Neighborhood Theorem 3.57, one can prove that $x \approx y$ if and only if $x \sim y$ and the isotropy representations of G_x on $T_x M$ and G_y on $T_y M$ descend to equivalent representations

[19] To our knowledge, the notion of *local orbit types* was introduced in Duistermaat and Kolk [79].

on $T_xM/T_xG(x)$ and on $T_yM/T_yG(y)$, that is, there exists an equivariant linear isomorphism between these spaces,[20] recall Exercise 3.7 and Proposition 3.41.

In what follows, we show that $M_x^{\approx}$ is a (possibly disconnected) embedded submanifold of M. As proved in [79, Theorem 2.6.7], $M_x^{\approx}$ is an open and closed subset of $M_x^{\sim}$. In particular, an orbit type $M_x^{\sim}$ is the disjoint union of submanifolds of possibly different dimensions, and connected components of the same dimension are arranged into the local orbit types $M_y^{\approx}$, for $y \in M_x^{\sim}$.

Orbit types $M_x^{\sim}$ are commonly denoted $M_{(H)}$ in the literature, in reference to the set of points with isotropy group conjugate to $H = G_x$. We prefer to use the notation $M_x^{\sim}$ (in association with $M_x^{\approx}$), in the same way as [79], to stress the dependence on the point $x \in M$ rather than on the conjugacy class of its isotropy group H. Indeed, conjugates of H may have different isotropy representations at different points. Furthermore, the notations $M_x^{\sim}$ and $M_x^{\approx}$ are also suggestive of leaves of a foliation.

Theorem 3.95. *Let $\mu\colon G \times M \to M$ be a proper action. Each local orbit type $M_x^{\approx}$ is a union of G-invariant embedded submanifolds (of the same dimension). Moreover, the restriction of the quotient map $\pi\colon M \to M/G$ to $M_x^{\approx}$ determines a fiber bundle*

$$G/H \to M_x^{\approx} \to M_x^{\approx}/G,$$

whose structure group is $N(H)/H$, where $H = G_x$ is the isotropy at x and $N(H)$ is the normalizer of H in G. In particular, subsets $M_x^{\approx}/G$ of orbits that have the same local orbit type are smooth manifolds inside M/G.

Proof (Sketch). The proof of this result is analogous to the proof of the Tubular Neighborhood Theorem 3.57, and details can be found in [79, pp. 109–111]. The main idea is to prove that $M_x^{\approx}$ is the total space of the associated bundle with fiber G/H,

$$G/H \to G/H \times_K P \to P/K, \tag{3.21}$$

to the principal K-bundle $K \to P \to P/K$, where $P := M_x^{\approx} \cap M^H$ and $K := N(H)/H$.

Using the properties in Definition 3.47 and the Tubular Neighborhood Theorem 3.57, one can prove the following, where $\mathrm{Tub}(G(x)) = \mu(G, S_x)$.

Claim 3.96. $M_x^{\sim} \cap \mathrm{Tub}(G(x))$ is a submanifold of M. Moreover, there is a G-equivariant diffeomorphism between $M_x^{\sim} \cap \mathrm{Tub}(G(x))$ and $G/H \times (S_x)^H$.

Note that if $y \in M_x^{\sim} \cap M^H$, then $G_y = gHg^{-1}$ for some $g \in G$ and $H \subset G_y$, and hence $H = G_y$ and $g \in N(H)$. Conversely, if $G_y = H$, then $y \in M_x^{\sim} \cap M^H$. Thus, we proved the first item in the following claim, whose other items are proved similarly.

[20] If the action is isometric, this simply means that the slice representations of G_x and G_y are equivalent, i.e., related by an equivariant linear map $\phi\colon \nu_x G(x) \to \nu_y G(y)$.

Claim 3.97. The following hold:

(i) $y \in M_x^{\sim} \cap M^H$ if and only if $G_y = H$.
(ii) Let $y \in M_x^{\sim} \cap M^H$. Then $\mu(g,y) \in M_x^{\sim} \cap M^H$ if and only if $g \in N(H)$;
(iii) $P = M_x^{\approx} \cap M^H$ is invariant under the subaction of $N(H)$, and the ineffective kernel of this action is H;
(iv) Each $y \in P$ has a neighborhood diffeomorphic to $(N(H)/H) \times (S_x)^H$.

Altogether, we have that P is a manifold with a free proper (left) K-action, so that $K \to P \to P/K$ is a principal K-bundle, and $P/K = M_x^{\approx}/G$. Moreover, it is not difficult to see that there is also a free (right) K-action on G/H (see Exercise 6.9), which allows us to construct the associated bundle (3.21) with fiber G/H.[21]

In order to prove that there exists a G-equivariant diffeomorphism between $M_x^{\approx}$ and the total space of this associated bundle, note that

$$\varphi\colon G/H \times P \to M_x^{\approx}, \quad \varphi(gH,s) := \mu(g,s),$$

is a well-defined map, by Claim 3.97. Given $y \in M_x^{\approx}$, we have $G_{\mu(g^{-1},y)} = H$ for some $g \in G$. From Claim 3.97, $\mu(g^{-1},y) \in M_x^{\sim} \cap M^H$. Since $M_x^{\approx}$ is G-invariant, $\mu(g^{-1},y) \in P$ and hence $y = \varphi\big(g,\mu(g^{-1},y)\big)$. Thus, φ is surjective. Furthermore, the restriction $\varphi\colon G/H \times (S_x)^H \to M_x^{\sim} \cap \mathrm{Tub}(G(x))$ is a diffeomorphism; and, since

$$T_{(gG_x,y)}\big(G/H \times (S_x)^H\big) \subset T_{(gH,y)}(G/H \times P), \tag{3.22}$$

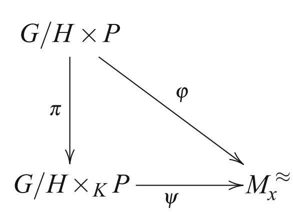
we conclude that $\mathrm{d}\varphi$ is surjective. We now observe the following (cf. Claim 3.59):

Claim 3.98. $\varphi(gH,y) = \varphi(hH,z)$ if and only if $h = gn^{-1}$ and $z = \mu(n,y)$ for some $n \in N(H)$.

Assume that $\varphi(gH,y) = \varphi(hH,z)$. Then $\mu(g,y) = \mu(h,z)$ and $z = \mu(n,y)$, where $n = h^{-1}g$. Since $z,y \in P$, it follows from Claim 3.97 that $n \in N(H)$. The converse statement is clear. We now define the candidate to G-equivariant diffeomorphism,

$$\psi\colon G/H \times_K P \to M_x^{\approx}, \quad \psi\big([gH,z]\big) := \mu(g,z),$$

cf. (3.16). The fact that $\pi\colon G/H \times P \to G/H \times_K P$, $\pi(gH,z) = [g,z]$, is the projection of a principal bundle, and Claim 3.98, imply that ψ is well-defined, smooth, bijective and G-equivariant. In only remains to prove that ψ is a diffeomorphism.



[21] Here, it is convenient to use a free *left* action on the principal bundle instead of a *right* action, cf. Proposition 3.33. Notice that, accordingly, the notation (3.21) for the twisted space is also reversed, cf. Theorem 3.51.

Note that $\mathrm{d}\pi$ and $\mathrm{d}\varphi$ are surjective, and $\varphi = \psi \circ \pi$, hence $\mathrm{d}\psi$ is surjective. By a dimension argument analogous to (3.17), it follows that $\mathrm{d}\psi$ is an isomorphism. Thus, by the Inverse Function Theorem, ψ is a local diffeomorphism, and hence a diffeomorphism, since it is bijective. $\square$

Remark 3.99. By Theorem 3.95, the result in Exercise 3.81 can be extended to the case of isometric actions that are not necessarily free. Namely, the projection map $\pi\colon M_x^{\approx} \to M_x^{\approx}/G$ is a Riemannian submersion, where $M_x^{\approx}/G$ is equipped with the natural orbit metric. Note that $M_x^{\approx}/G$ is *locally totally geodesic* in M/G, since by Claim 3.96, it is locally isometric to the fixed point set $(S_x)^H$, where S_x is a slice at $x \in M$, which is totally geodesic by Proposition 3.93.

We conclude this section with a discussion of the *orbit type stratification*.

Definition 3.100. A *stratification* of a manifold M is a partition of M by embedded submanifolds $\{M_i\}_{i\in I}$ of M, called *strata*, such that:

(i) The partition is *locally finite*, i.e., each compact subset of M only intersects a finite number of strata;
(ii) For each $i \in I$, there exists a subset $I_i \subset I \setminus \{i\}$ such that the closure of M_i is $\overline{M_i} = M_i \cup \bigcup_{j\in I_i} M_j$;
(iii) $\dim M_j < \dim M_i$, for all $j \in I_i$.

Example 3.101. Consider the action of $\mathrm{SO}(2)$ on $\mathbb{R}^3$ described in Example 3.4 (ii). Then $\{M_i\}_{i\in I}$, given by

$$M_1 := \big\{(0,0,x_3) \in \mathbb{R}^3 : x_3 \in \mathbb{R}\big\}, \quad M_2 := \mathbb{R}^3 \setminus M_1, \tag{3.23}$$

is a stratification of $\mathbb{R}^3$, where $I = \{1,2\}$, $I_1 = \emptyset$ and $I_2 = \{1\}$. The stratum M_1 consists of fixed points (which are singular orbits), and the stratum M_2 consists of regular points (which lie in principal orbits). Note that $M_2 = M_{\mathrm{princ}}$ is open an dense and M_{princ}/G is connected, as asserted by the Principal Orbit Theorem 3.82.

Theorem 3.102. *Let $\mu\colon G \times M \to M$ be a proper action. The partition of M into connected components of the orbit types $M_x^{\sim}$, $x \in M$, determines a stratification.*

Proof. The partition of M into connected components of orbit types is locally finite by the same arguments from the proof of Theorem 3.91. Given $y \in M$, we denote by $(M_y^{\sim})^0$ the connected component of $M_y^{\sim}$ that contains y.

Claim 3.103. Let $x \in \overline{(M_y^{\sim})^0}$ be such that $x \notin (M_y^{\sim})^0$. Then:

(i) $(M_x^{\sim})^0$ is contained in $\overline{(M_y^{\sim})^0}$;
(ii) $\dim(M_x^{\sim})^0 < \dim(M_y^{\sim})^0$.

In order to prove (i), we may assume that $y \in S_x$. From Claim 3.96, it suffices to prove that if $\widetilde{x} \in (S_x)^H$, where $H = G_x$, then $\widetilde{x} \in \overline{(M_y^{\sim})^0}$. Using the slice representation, we may also assume that $(S_x)^H$ is a linear subspace fixed by the linear action of $K = \mu^H$ on a vector space, and $x = 0$. Since the K-action fixes each point of $(S_x)^H$, it leaves invariant the normal space $\nu_p(S_x)^H$ that contains y. Consider the projection $\widetilde{y} = y - p + \widetilde{x}$ of y onto the normal space $\nu_{\widetilde{x}}(S_x)^H$. As $K_{\widetilde{x}} = K = K_p$, we have that $K_y = K_{\widetilde{y}}$. Moreover, as the K-action is linear, $K_z = K_{\widetilde{y}}$ for each z different from $\widetilde{x}$ along the straight line segment joining $\widetilde{x}$ to $\widetilde{y}$. Thus, $\widetilde{x} \in \overline{(M_y^\sim)^0}$, concluding the proof of (i).

As for (ii), let $(S_x)_y^\sim$ be the set of points in S_x that have the same H-orbit type of $y \in S_x$. From the above discussion, we infer that $\dim((S_x)_y^\sim)^0 \geq \dim(S_x)^H + \dim \mu^H(y) + 1$. In particular, $\dim((S_x)_y^\sim)^0 > \dim(S_x)^H$. Thus, using Claim 3.96 and the fact that each point in $((S_x)_y^\sim)^0$ is also in $(M_y^\sim)^0$, we have that

$$\dim(M_x^\sim)^0 = \dim G(x) + \dim(S_x)^H < \dim G(x) + \dim((S_x)_y^\sim)^0 \leq \dim(M_y^\sim)^0,$$

which concludes the proof of (ii). □

Remark 3.104. Furthermore, it is possible to prove that an orbit type stratification is a *Whitney stratification*, see Duistermaat and Kolk [79].

Remark 3.105. Each stratum of the orbit type stratification of an isometric action is a *minimal submanifold*. In fact, the mean curvature vector of a stratum (see (2.9)) is tangent to the stratum by equivariance, but is also normal to the stratum by definition, and hence vanishes.

Remark 3.106. The orbit type stratification $\{M_i\}_{i\in I}$ of a manifold M induces a stratification $\{M_i/G\}_{i\in I}$ of the corresponding orbit space M/G, i.e., a partition into manifolds satisfying (i)–(iii) in Definition 3.100. From Remark 3.99, the orbit strata are locally totally geodesic. This plays an important role when M/G is studied from the viewpoint of Alexandrov geometry, see Remark 3.80.

Example 3.107. Consider the conjugation action of $\mathrm{SU}(3)$ on itself. From Exercise 3.88, there are 3 orbit types, since an orbit is either a fixed point, the complex projective plane $\mathbb{C}P^2$, or the complex flag manifold $\mathrm{SU}(3)/T$. Clearly, the latter are the principal orbits, and it is not difficult to show that each of the 2 nonprincipal orbit types have 3 connected components (of the same dimension). This describes the orbit type stratification of $\mathrm{SU}(3)$, see Theorem 3.102. The orbit space of this conjugation action is a flat equilateral triangle, whose induced stratification is the obvious partition of a closed triangle into the union of its vertices (corresponding to fixed points in $\mathrm{SU}(3)$), edges (corresponding to the orbits diffeomorphic to $\mathbb{C}P^2$), and interior (corresponding to principal orbits).

The linearization of the conjugation action of $\mathrm{SU}(3)$ at the identity $I \in \mathrm{SU}(3)$ is its adjoint representation, see (1.8). Notice that this is precisely the isotropy representation (and the slice representation) at $I \in \mathrm{SU}(3)$, since I is a fixed point of conjugation. The orbit structure of this adjoint action is studied in Example 4.42, where it is proved that its orbit space is a (closed) wedge of angle $\pi/3$ in the plane. We remark that this is precisely the so-called *tangent cone* to the orbit space of the conjugation action (the equilateral triangle) at the identity orbit (one of the vertices).

This is an instance of a general relation between tangent cones at points of the orbit space of an isometric action and the orbit space of the corresponding slice representation.

Remark 3.108. As demonstrated many times along this chapter (e.g., Theorems 3.82, 3.91 and 3.102), the local study of proper (or isometric) actions can be reduced to the local study of an isometric action in Euclidean space, by using the slice representation. Molino [165] pointed out that the same idea can be used in the local study of singular Riemannian foliations (see Definition 5.2). More precisely, after a suitable change of metrics, the local study of any singular Riemannian foliation is reduced to the study of a singular Riemannian foliation on the Euclidean space (with the standard metric). This idea is used, for example, in [8, 20], see also the survey [12].