Problems in Comparison Geometry

In all problems below, (M, g) is a complete smooth Riemannian manifold, and S^n_k denotes the n-dimensional round sphere of radius $\frac{1}{\sqrt{k}}$, which is simply denoted S^n if $k = 1$.

Problems related to Bishop-Gromov relative volume comparison

1. **Cheng's Theorem (Rigidity in Bonnet-Myers).** If (M^n, g) has $\mathrm{Ric} \geq (n-1)k > 0$ and $\mathrm{diam}(M, \mathrm{g}) = \frac{\pi}{\sqrt{k}}$, then (M^n, g) is isometric to the round sphere S^n_k.

2. **Corollary to Grove-Shiohama Diameter Sphere Theorem.** Show that if (M^n, g) has $\sec \geq k > 0$ and $\mathrm{Vol}(M, \mathrm{g}) > \frac{1}{2}\mathrm{Vol}(S^n_k)$, then $\mathrm{diam}(M, \mathrm{g}) > \frac{1}{2}\mathrm{diam}(S^n_k)$ and hence M is homeomorphic to a sphere.

3. **Towards a 'Volume Sphere Theorem'.** Show that if (M^n, g) has $\mathrm{Ric} \geq (n-1)k > 0$ and $\mathrm{Vol}(M, \mathrm{g}) > \frac{1}{2}\mathrm{Vol}(S^n_k)$, then M is simply-connected.

   Note: It is a well-known conjecture that such a manifold (M^n, g) should be homeomorphic (or even diffeomorphic) to a sphere. This would be a *Ricci curvature* analogue of the Grove-Shiohama Diameter Sphere Theorem.

4. Show that the statement above becomes false if $\mathrm{Vol}(M, \mathrm{g}) > \frac{1}{2}\mathrm{Vol}(S^n_k)$ is relaxed to $\mathrm{Vol}(M, \mathrm{g}) \geq \frac{1}{2}\mathrm{Vol}(S^n_k)$, by exhibiting an example.

   More context: Perelman showed that if (M^n, g) has $\mathrm{Ric} \geq (n-1)$ and $\mathrm{Vol}(M, \mathrm{g}) \geq (1 - \delta_n)\mathrm{Vol}(S^n)$, then M is homeomorphic to S^n. Anderson constructed examples of metrics on $\mathbb{C}P^n$, among many other manifolds, with $\mathrm{Ric} \geq (n-1)$ and $\mathrm{diam}(M, \mathrm{g}) \geq \pi - \varepsilon$, showing that the lower volume bound in the conjecture cannot be relaxed to a lower diameter bound.

Problems related to Toponogov comparison

5. **Toponogov's original theorem.** Let (M^2, g) be a surface with $\sec \geq k > 0$. Any simple closed geodesic γ on (M, g) has length $l(\gamma) \leq \frac{2\pi}{\sqrt{k}}$. If equality holds for any such geodesic, then (M^2, g) is isometric to the round sphere S^n_k.

6. **First step in proving the Grove-Shiohama Diameter Sphere Theorem**. Show that if (M, g) has $\sec \geq k > 0$ and $\mathrm{diam}(M, \mathrm{g}) > \frac{\pi}{2\sqrt{k}}$, then given any point $p \in M$, there exists a *unique* point $q \in M$ with $\mathrm{dist}(p, q) = \mathrm{diam}(M, \mathrm{g})$.