Writhe - Total Curvature.05 for Grad Student Geom-Top Seminar

September 9, 2017

TOTAL CURVATURE: A few sample results plus references

The curvature κ of a smooth curve in 3-space is ≥ 0 by defInition, and its integral w.r.t. arc length, $\int \kappa(s)\, ds$, is called the ***total curvature*** of the curve.

In the plane, the curvature κ can change sign, so we define total curvature to be $\int |\kappa(s)|\, ds$, and sometimes, for emphasis, this is called total absolute curvature.

The two basic theorems in the subject.

Fenchel's Theorem (1929). The total curvature of any simple closed curve in 3-space is $\geq 2\pi$, with equality if and only if it is a plane convex curve.

Fary-Milnor Theorem (1949). If the simple closed curve in 3-space is knotted, then its total curvature is $> 4\pi$.

Surfaces in 4-space.

Theorem (Dan Sunday, 1974). If $f : S^2 \to R^4$ is a smooth embedding, and if gen(f) is the minimum number of generators needed to present the fundamental group $\pi_1(R^4 - f(S^2))$, then

$$\text{Total curvature of } f(S^2) \geq 4\pi^2 \,\text{gen}(f).$$

The argument uses Morse theory in much the same way that it can be used to give a proof of the Fary-Milnor Theorem.

Two results due to Peter Wintgen in 1978.

(A) Let f: $S^2 \to R^4$ be an immersion, let H be the mean curvature of the image, and I the self-intersection number in the sense of Whitney. Then the total squared mean curvature satisfies

$$\int_{S^2} H^2 \, d(\text{area}) \geq (3 + I)\,\pi \,.$$

(B) Let f: $M^2 \to R^4$ be a smooth embedding of a closed surface M^2 into R^4, and gen(f) the minimum number of generators of the fundamental group $\pi_2(R^4 - f(S^2))$. Then

$$\int_{M^2} H^2 \, d(\text{area}) \geq 4\,\pi\,\text{gen}(f) \,.$$

Compare Wintgen's results w. the Willmore conjecture.

If M^2 is a smooth closed orientable surface in R^3 (***not*** R^4), then $\int_{M^2} H^2 \, d(\text{area})$, where H is the mean curvature of M^2, is known as the ***Willmore energy*** $W(M^2)$ of this surface.

One can prove that $W(M^2) \geq 4\pi$, with equality if and only if M^2 is a round sphere.

Compare with Wintgen's inequality $\int_{S^2} H^2 \, d(\text{area}) \geq (3 + I)\,\pi$ for 2-spheres immersed in R^4.

In 1965, Willmore, proposed that if M^2 is a smoothly immersed torus in R^3, then $W(M^2) \geq 2\pi^2$.

This was proved in 2012 by Fernando Coda Marques and André Neves, using the Almgren-Pitts min-max theory.

Compare with Wintgen's inequality $\int_{M^2} H^2 \, d(\text{area}) \geq 4\,\pi\,\text{gen}(f)$ for surfaces smoothly embedded in R^4.

Theorem of Remi Langevin and Harold Rosenberg (1976) .

Let C be a smooth knot embedded in R^3 and M a smooth compact orientable surface of genus g immersed in R^3 with boundary C . Then

$$2 \int_C \kappa \, ds + \int_M |K| \, d(\text{area}) \geq 2\pi (2g + 2) ,$$

where κ is the curvature (≥ 0) of C and K is the Gaussian curvature of M , and if C is actually knotted, then the inequality is $\geq 2\pi (2g + 3)$.

Theorems of Chern and Lashof (1957).

Let M^n be a closed orientable n-manifold immersed in R^N . Then

(1) Total curvature of $M^n \geq 2 \operatorname{vol}(S^{N-1})$, with equality if and only if M^n is embedded as a convex hypersurface in some n+1 plane in R^N .

(2) If Total curvature of $M^n < 3 \operatorname{vol}(S^{N-1})$, then M is homeomorphic to S^{n-1} .

Fundamental question to study, as phrased by Joe Hoisington.

Let M^n be a closed n-manifold, smoothly embedded in R^N .

How does the "topological complexity" of the manifold M^n and its complement $R^N - M^n$ impose a lower bound on the total curvature of M^n ?

(1) Study first for knots in 3-space ...

(2) then for surfaces in 4-space.

TOTAL CURVATURE REFERENCES

1929 Werner Fenchel, *Uber Krummung und Windung geschlossener Raumkurven,* Math. Ann. 101, 238 - 252.

1949 Wilhelm Blaschke, *Sulla geometria differenziale delle superficie S_2 nello spazio euclideo S_4 ,* Annali di Matematica, Serie IV, Vol 28, 205 - 209.

1949 Istvan Fáry, *Sur la courbure totale d'une courbe gauche faisant un noeud,* Bull. Soc. Math. France, 77, 128 - 138.

1950 John Milnor, *On the total curvature of knots*, Annals of Math. 52 No. 2, 248 - 257.

1950 Ralph Fox, *On the total curvature of some tame knots,* Annals of Math. 52 No. 2, 258 - 260.

1951 Werner Fenchel, *On the differential geometry of closed space curves,* Bull. AMS 57, 44 - 54.

1951 Shiing-shen Chern and Edwin Spanier, *A theorem on orientable surfaces in 4-dimensional space,* Comm. Math. Helv. 25, 205 - 209.

1953 John Milnor, *On total curvatures of closed space curves,* Math. Scand. 1, 289 - 296.

1955 Shiing-shen Chern, *La geometrie des sous-varietes d'un espace euclidien a plusiers dimensions,* L'Enseigement Math. 40, 26 - 46.

1957 Shiing-shen Chern and Richard Lashof, *On the total curvature of immersed manifolds,* Amer. J. Math. 79, 306 - 318.

1957 Shiing-shen Chern and Richard Lashof, *On the total curvature of immersed manifolds II,* Michigan Math. J. 5, 5 - 12.

1958 Richard Lashof and Steven Smale, *On the immersion of manifolds in Euclidean space,* Ann. of Math. (2) 68, 562 - 583.

1967 D. Ferus, *Uber die absolut Totalkrummung hoherdimensionaler Knoten,* Math. Ann. 71, 81 - 86.

1974 Dan Sunday, *The total curvature of knotted spheres,* Bull. AMS, 82, No. 1, 140 - 142.

1976 Remi Langevin and Harold Rosenberg, *On curvature integrals and knots,* Topology 15, 405 - 416.

1978 Peter Wintgen, *On the total curvature of surfaces in* E^4*, Colloq. Math. 39, No. 2, 289 - 296.*

1978 Peter Wintgen, *On the total absolute curvature of knotted spheres,* Bull. Acad. Polon. Ser. Sci. Math. Astron. Phys 26, No. 3, 249 - 253.

2014 Fernando Coda Marques and André Neves, *Min-max theory and the Willmore conjecture,* Annals of Math. 179, 683-782.