Math 241, Fall 2015 **Renato Ghini Bettiol**

Homework Set 3

Due: Sep 22, 2015 (in class)

1. Let $u(x, y)$ be the solution to the Laplace equation $\Delta u = 0$ on the square $0 < x < 1$, $0 < y < 1$, with boundary values $+1$ on the top and bottom sides and -1 on the left and right sides, that is, $u(x, 0) = u(x, 1) = 1$ and $u(0, y) = u(1, y) = -1$. The goal of this problem is to compute $u\left(\frac{1}{3}, \frac{1}{3}\right)$ without knowing an explicit formula for $u(x, y)$, but instead using the *reflection symmetry* about the diagonal line $x = y$.
   a) Define $v(x, y) = u(y, x)$. Show that $\Delta v = 0$, i.e., v also solves the Laplace equation.
   b) Compute the boundary values of $v(x, y)$ on the square $0 < x < 1$, $0 < y < 1$.
   c) Define $w(x, y) = u(x, y) + v(x, y)$. Compute the boundary values of $w(x, y)$ on the square $0 < x < 1$, $0 < y < 1$. Use the fact that solutions to the Laplace equation with given boundary values are unique to conclude that $w \equiv 0$ and hence $u(x, y) = -v(x, y) = -u(y, x)$. Compute $u\left(\frac{1}{3}, \frac{1}{3}\right)$.

2. Haberman 2.5.1 (a), (b), (c), (d)

3. Haberman 2.5.3

4. Haberman 2.5.5 (a), (b), (c)

5. Using separation of variables, solve the wave equation $u_{tt} = c^2 u_{xx}$ for $0 < x < L$ with the following boundary conditions and initial conditions:
   a) $u(0, t) = 0$, $u(L, t) = 0$, $u(x, 0) = 0$, $u_t(x, 0) = 3 \sin \frac{3\pi x}{L}$.
   b) $u(0, t) = 0$, $u(L, t) = 0$, $u(x, 0) = 3 \sin \frac{3\pi x}{L}$, $u_t(x, 0) = 0$.
   c) $u(0, t) = 0$, $u(L, t) = 0$, $u(x, 0) = \sin \frac{2\pi x}{L} + 7 \sin \frac{5\pi x}{L}$, $u_t(x, 0) = 0$.
   d) $u(0, t) = 0$, $u(L, t) = 0$, $u(x, 0) = \sin \frac{2\pi x}{L} + 7 \sin \frac{5\pi x}{L}$, $u_t(x, 0) = 2 \sin \frac{3\pi x}{L} + 4 \sin \frac{6\pi x}{L}$.

6. Using the d'Alembert solution, solve the wave equation $u_{tt} = c^2 u_{xx}$ for $-\infty < x < +\infty$ with the following initial conditions:
   a) $u(x, 0) = 0$, $u_t(x, 0) = 1$.
   b) $u(x, 0) = 0$, $u_t(x, 0) = x^2$.
   c) $u(x, 0) = \sin x$, $u_t(x, 0) = x^2$.
   d) $u(x, 0) = \ln(1 + x^2)$, $u_t(x, 0) = 2$.