Math 241, Fall 2015 **Renato Ghini Bettiol**

Homework Set 9

DUE: NOV 24, 2015 (IN CLASS)

1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, and denote by $\vec{n}$ the outer unit normal to $\partial\Omega$. Prove that the nonhomogeneous Neumann problem

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ \langle \nabla u, \vec{n} \rangle = 0 & \text{on } \partial\Omega. \end{cases}$$

   has a solution *only if* $\int_\Omega f = 0$. [REMARK. This *"only if"* is actually *"if and only if"*.]

2. Consider the nonhomogeneous Poisson equation $u''(x) = 1$ on the interval $0 < x < 1$, with Dirichlet boundary conditions $u(0) = u(1) = 0$.
   a) Find an explicit formula for the solution $u(x)$.
   b) Compute the coefficients of the Fourier Sine Series of the function $f(x) \equiv 1$, and use the method of eigenfunction expansion to find $u(x)$ as a Fourier Sine Series.
   c) Verify that the series you obtained in b) is the Fourier Sine Series of the function you computed in a).

3. Let D be the unit disk in $\mathbb{R}^2$, and consider the nonhomogeneous Poisson problem

$$\begin{cases} \Delta u = 4 & \text{in } D, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

   where we assume that $|u(0, \theta)| < +\infty$.
   a) Use the method of eigenfunction expansion to find a solution $u(r, \theta)$ in terms of the Fourier-Bessel Series of the constant function $f(r, \theta) \equiv 4$.
   b) Since $f(r, \theta)$ is *radial*, i.e., independent of θ, we know that $u(r, \theta)$ must also be radial. Use this to find an explicit formula for $u(r, \theta) = u(r)$. Conclude that the series you obtained in a) is the Fourier-Bessel Series of $u(r)$.

4. Haberman 8.6.3 (a), (b), (c)

5. Haberman 8.6.6

6. Haberman 8.6.7

7. Prove the *frequency shift property*; that is, if $g(x) = e^{ix\xi_0} f(x)$, then $\hat{g}(\xi) = \hat{f}(\xi - \xi_0)$.

8. Prove the *time scaling property*; that is, if $g(x) = f(ax)$, then $\hat{g}(\xi) = \frac{1}{|a|} \hat{f}\left(\frac{\xi}{a}\right)$.

9. Haberman 10.3.6