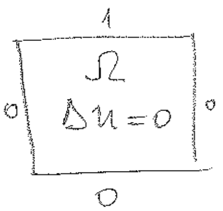
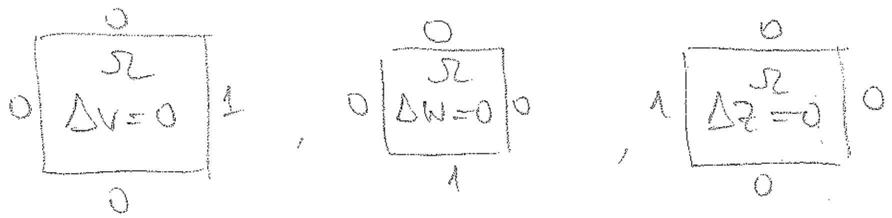
Practice problems for Midterm 1

#1

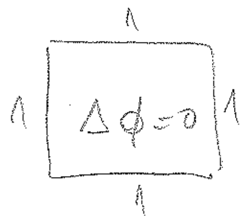


$\Omega = [0,1] \times [0,1]$

a) Let $v, w, z : \Omega \to \mathbb{R}$ be solutions to the following boundary value problems for the Laplace equation:



By uniqueness of solutions, it follows that v, w, z can be obtained from u by composing with a rotation of the square Ω by $\frac{\pi}{2}$, π and $\frac{3\pi}{2}$ resp. about the central point $\left(\frac{1}{2}, \frac{1}{2}\right) \in \Omega$. In particular, $u\left(\frac{1}{2}, \frac{1}{2}\right) = v\left(\frac{1}{2}, \frac{1}{2}\right) = w\left(\frac{1}{2}, \frac{1}{2}\right) = z\left(\frac{1}{2}, \frac{1}{2}\right)$. Moreover, by linearity (or the "principle of superposition"), we have that $\phi = u + v + w + z$ satisfies



$$\begin{cases} \Delta \phi = 0 & \text{in } \Omega \\ \phi = 1 & \text{on } \partial\Omega \end{cases}$$

By uniqueness of solutions, $\phi \equiv 1$ is a constant function. Thus,

$$\begin{aligned} 1 = \phi\left(\tfrac{1}{2}, \tfrac{1}{2}\right) &= u\left(\tfrac{1}{2}, \tfrac{1}{2}\right) + v\left(\tfrac{1}{2}, \tfrac{1}{2}\right) + w\left(\tfrac{1}{2}, \tfrac{1}{2}\right) + z\left(\tfrac{1}{2}, \tfrac{1}{2}\right) \\ &= 4\, u\left(\tfrac{1}{2}, \tfrac{1}{2}\right). \end{aligned}$$

Therefore $\boxed{u\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \tfrac{1}{4}}$

b) By the maximum principle,

$$\max_{(x,y)\in\bar\Omega} u(x,y) = \max_{(x,y)\in\partial\Omega} u(x,y) = 1 \quad \text{and} \quad \min_{(x,y)\in\Omega} u(x,y) = \min_{(x,y)\in\partial\Omega} u(x,y) = 0.$$

are attained at $\partial\Omega$.

#3 $\quad u_t = u_{xx} + 4 \qquad 0 < x < 1$

(BC) $\quad u_x(0,t) = 5 \qquad$ (IC) $\quad u(x,0) = f(x)$.

$\qquad u_x(1,t) = \beta$

$$H(t) = \int_0^1 u(x,t)\,dx$$

$$H'(t) = \frac{d}{dt}\int_0^1 u(x,t)\,dx = \int_0^1 u_t(x,t)\,dx = \int_0^1 \left(u_{xx}(x,t) + 4\right)dx$$

$$= u_x(1,t) - u_x(0,t) + 4x\Big|_0^1$$

$$= \beta - 5 + 4 = \beta - 1$$

thus, $\quad H(t) = (\beta-1)t + H(0)$

where $\quad H(0) = \int_0^1 u(x,0)\,dx = \int_0^1 f(x)\,dx$.

So $\quad \boxed{H(t) = (\beta-1)t + \int_0^1 f(x)\,dx}$

If the solution is an equilibrium, then $H(t) \equiv$ const., so we must have $\beta = 1$. (Let us see this in another way next) $\longrightarrow$

#2 $u_t = u_{xx} + 4$

$u_x(0,t) = 5$

$u_x(1,t) = \beta$

If there is an equilibrium solution, then:

$u_t(x,t) = 0 \quad \Rightarrow \quad u_{xx}(x,t) + 4 = 0$

$\Rightarrow \quad u_x(x,t) = -4x + c$

$\Rightarrow \quad u(x,t) = -2x^2 + cx + d.$

From $u_x(0,t) = 5$, we get $u_x(0,t) = \underline{\underline{c = 5}}$.

From $u_x(1,t) = \beta$, it follows that

$\beta = u_x(1,t) = -4 + c = \underline{1}$

Thus $\beta = \underline{1}$ is a <u>necessary</u> condition for the existence of an equilibrium solution, which, up to the constant $d \in \mathbb{R}$, is given by

$$u(x,t) = -2x^2 + 5x + d$$

<u>Remark:</u> If we were given an (IC) $u(x,0) = f(x)$, then we could compute d in terms of $\int_0^1 f(x)\,dx$

$$\int_0^1 u(x,t)\,dx = \int_0^1 f(x)\,dx \Rightarrow \left(-\frac{2}{3}x^3 + 5\frac{x^2}{2} + dx\right)\Big|_0^1 = \int_0^1 f(x)\,dx$$

$$\Rightarrow -\frac{2}{3} + \frac{5}{2} + d = \int_0^1 f(x)\,dx \Rightarrow \boxed{d = -\frac{11}{6} + \int_0^1 f(x)\,dx}$$

#4. $u_t = 3u_{xx}, \quad 0 < x < 1$

$u(0,t) = u(1,t) = 0$ $\qquad \begin{pmatrix} L=1 \\ k=3 \end{pmatrix}$

$u(x,0) = 2\sin(\pi x) + 5\sin(4\pi x)$

By separation of variables, the general solution to the heat equation with Dirichlet boundary conditions is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\, e^{-3(n\pi)^2 t}$$

Matching the initial condition, we get:

$$\boxed{u(x,t) = 2\sin(\pi x)\, e^{-3\pi^2 t} + 5\sin(4\pi x)\, e^{-3\cdot(4\pi)^2 t}}$$

#5 $u_t = 3u_{xx}, \quad 0 < x < 1$

$u_x(0,t) = u_x(1,t) = 0$ $\qquad \begin{pmatrix} L=1 \\ k=3 \end{pmatrix}$

$u(x,0) = 2 + 3\cos(4\pi x)$

By separation of variables, the general solution to the heat equation with Neumann boundary conditions is:

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x)\, e^{-3(n\pi)^2 t}$$

Matching the initial conditions, we get:

$$\boxed{u(x,t) = 2 + 3\cos(4\pi x)\, e^{-3(4\pi)^2 t}}$$

#6 $u_{tt} = 4u_{xx} \quad 0<x<1$

$u(0,t) = u(1,t) = 0$ $\begin{pmatrix} L=1 \\ c=2 \end{pmatrix}$

$u(x,0) = 2\sin(3\pi x) = f(x)$

$u_t(x,0) = 6\sin(9\pi x) = g(x)$

By separation of variables, the general solution to the wave equation with Dirichlet conditions is

$$u(x,t) = \sum_{n=1}^{\infty} \Big(A_n \cos(2n\pi t) + B_n \sin(2n\pi t) \Big) \sin(n\pi x)$$

Matching the initial conditions, we get:

$$A_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx \implies A_3 = 2, \text{ all other } A_n = 0$$

$$B_n = \frac{2}{2n\pi}\int_0^1 g(x)\sin(n\pi x)\,dx \implies B_9 = \frac{1}{9\pi}\cdot 6\cdot\frac{1}{2} = \frac{1}{3\pi}$$

Thus $\boxed{u(x,t) = 2\cos(6\pi t)\sin(3\pi x) + \frac{1}{3\pi}\sin(18\pi t)\sin(9\pi x)}$

#7 $u_{tt} = 4u_{xx}, \quad 0<x<1$ $\begin{pmatrix} L=1 \\ c=2 \end{pmatrix}$

$u_x(0,t) = u_x(1,t) = 0$

$u(x,0) = 1 + 2\cos(3\pi x) = f(x)$

$u_t(x,0) = 2 + 6\cos(9\pi x) = g(x)$

By separation of variables, the general solution to the wave equation with Neumann conditions is

$$u(x,t) = A_0 + B_0 t + \sum_{n=1}^{\infty} \Big(A_n \cos(2n\pi t) + B_n \sin(2n\pi t) \Big) \cos(n\pi x)$$

Matching the initial conditions, we get:

$A_0 = 1$ $\quad A_3 = 2$ $\quad$ All other A_n and B_n vanish.

$B_0 = 2$ $\quad B_9 = \frac{1}{2 \cdot 9\pi} \cdot 6 = \frac{1}{3\pi}$

thus,

$$\boxed{u(x,t) = 1 + 2t + 2\cos(6\pi t)\cos(3\pi x) + \frac{1}{3\pi}\sin(18\pi t)\cos(9\pi x)}$$

#8 $\quad u_{tt} = 4u_{xx} \quad -\infty < x < \infty \qquad (c=2)$

$u(x,0) = 2\sin(3\pi x) = f(x)$

$u_t(x,0) = 6\sin(9\pi x) = g(x)$

By the d'Alembert solution,

$$u(x,t) = \frac{1}{2}\left(f(x+2t) + f(x-2t)\right) + \frac{1}{4}\int_{x-2t}^{x+2t} g(y)\,dy$$

So

$$u(x,t) = \sin(3\pi(x+2t)) + \sin(3\pi(x-2t)) + \frac{3}{2}\int_{x-2t}^{x+2t} \sin(9\pi y)\,dy$$

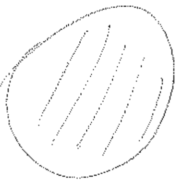
$$= \sin(3\pi x + 6\pi t) + \sin(3\pi x - 6\pi t) - \frac{3}{2}\left(\frac{1}{9\pi}\cos(9\pi y)\Big|_{x-2t}^{x+2t}\right)$$

$$= \sin(3\pi x + 6\pi t) + \sin(3\pi x - 6\pi t) - \frac{1}{6\pi}\left(\cos(9\pi x + 18\pi t) - \cos(9\pi x - 18\pi t)\right)$$

this answer is already fine, no need to simplify further...

$$\boxed{u(x,t) = 2\sin(3\pi x)\cos(6\pi t) + \frac{1}{3\pi}\sin(9\pi x)\sin(18\pi t),}$$

Remark: Compare with the solution to #6.

#9 $\Delta u = 0$ on disk of radius 1 

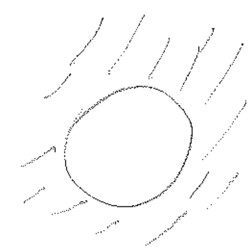
$u(1,\theta) = 7 + 9\cos\theta, \qquad |u(0,\theta)| < \infty$

By separation of variables, the general solution is

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} B_n r^n \sin(n\theta)$$

Matching the boundary condition, $A_0 = 7$, $A_1 = 9$, all other A_n and B_n vanish. Thus,

$$\boxed{u(r,\theta) = 7 + 9r\cos\theta}$$

#10 $\Delta u = 0$ on <u>outside</u> of disk of radius 1 

$u(1,\theta) = 7 + 9\cos\theta, \qquad \lim_{r\to\infty} |u(r,\theta)| < \infty.$

By separation of variables, the general solution is:

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n r^{-n} \cos(n\theta) + \sum_{n=1}^{\infty} B_n r^{-n} \sin(n\theta)$$

Matching the boundary condition, $A_0 = 7$, $A_1 = 9$, all other A_n and B_n vanish. Thus,

$$\boxed{u(r,\theta) = 7 + \frac{9}{r}\cos\theta}$$

<u>Remark</u>: Notice that the max and min of $u(r,\theta)$ on both #9 and #10 are resp. 16 and -2, and achieved (only) at the boundary $(r=1)$, as predicted by the <u>maximum</u> <u>principle.</u> Also, $u(0,\theta) = 7$ on #9, and the average value of $u(1,\theta) = 7 + \cos 9\theta$ is exactly 7, cf. <u>Mean value property.</u>

\# 11

$$\begin{cases} X''''(x) = \lambda X(x), & \lambda > 0 \\ X(0) = X(L) = 0 \\ X''(0) = X''(L) = 0 \end{cases}$$

the roots of the assoc. polynomial are $\pm\sqrt[4]{\lambda}$ and $\pm i\sqrt[4]{\lambda}$.
Let us call $\alpha = \sqrt[4]{\lambda}$, so these are $\pm\alpha$ and $\pm i\alpha$.
Then the basic solutions to the ODE $X'''' = \lambda X$ are

$$X_1(x) = e^{\alpha x},\ X_2(x) = e^{-\alpha x},\ X_3(x) = \cos(\alpha x),\ X_4(x) = \sin(\alpha x)$$

Thus the general solution is:

$$X(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x} + c_3 \cos(\alpha x) + c_4 \sin(\alpha x)$$

From $X(0) = 0$ and $X''(0) = 0$ we get:

$$0 = X(0) = c_1 + c_2 + c_3$$

$$0 = X''(0) = \alpha^2 c_1 + \alpha^2 c_2 - \alpha^2 c_3 = \alpha^2 (c_1 + c_2 - c_3)$$

thus $c_3 = c_1 + c_2 = -c_3$, so $\boxed{c_3 = 0}$ and $\boxed{c_1 + c_2 = 0}$

From $X(L) = 0$ and $X''(L) = 0$, we get:

$$0 = X(L) = c_1 e^{\alpha L} - c_1 e^{-\alpha L} + c_4 \sin(\alpha L)$$

$$0 = X''(L) = \alpha^2 c_1 e^{\alpha L} - \alpha^2 c_1 e^{-\alpha L} - \alpha^2 c_4 \sin(\alpha L) = \alpha^2 (c_1 e^{\alpha L} - c_1 e^{-\alpha L} - c_4 \sin(\alpha L))$$

Thus $c_4 \sin(\alpha L) = c_1 e^{\alpha L} - c_1 e^{-\alpha L} = -c_4 \sin(\alpha L)$

thus $c_4 \sin(\alpha L) = 0$ and $c_1 \underbrace{(e^{\alpha L} - e^{-\alpha L})}_{\sinh(\alpha L)} = 0$, so since

$\alpha \neq 0$ and $L \neq 0$, it follows that $\boxed{c_1 = 0}$ and $\boxed{\alpha L = n\pi}$ for some $n \in \mathbb{Z}$, Thus $\alpha = \frac{n\pi}{L}$, which means that $\boxed{\lambda = \left(\frac{n\pi}{L}\right)^4}$,

Moreover, $c_1 = 0$, hence $c_2 = 0$, and also we had $c_3 = 0$.

So the eigenvalues and eigenfunctions of this problem are:

$$\boxed{\lambda_n = \left(\frac{n\pi}{L}\right)^4, \quad \phi_n(x) = \sin\left(\frac{n\pi x}{L}\right) \quad n = 1, 2, 3, \ldots}$$

#12 $\quad u_t + u_{xxxx} = 0, \quad 0 < x < l$

(BC) $\quad u(0,t) = u(L,t) = 0$
$\quad u_{xx}(0,t) = u_{xx}(L,t) = 0.$

To apply separation of variables, we look for solutions of the form $u(x,t) = X(x)T(t)$, the PDE then becomes

$$XT' + X''''T = 0 \Rightarrow \underbrace{\frac{T'}{T}}_{\text{function of } t \text{ only}} = -\underbrace{\frac{X''''}{X}}_{\text{function of } x \text{ only}} = -\lambda \leftarrow \text{separation constant.}$$

$$\begin{cases} X'''' = \lambda X \\ T' = -\lambda T \end{cases}$$

The (BC) give that either $T \equiv 0$ or $\begin{cases} X(0) = X(L) = 0 \\ X''(0) = X''(L) = 0. \end{cases}$

Thus, we have the eigenvalue problem from #11, whose eigenvalues and eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^4, \quad \phi_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots$$

the solution of the equation for $T(t)$ is

$$T(t) = ce^{-\lambda t} = ce^{-\left(\frac{n\pi}{L}\right)^4 t}$$

Therefore, we get solutions to our original problem:

$$u_n(x,t) = X_n(x)T_n(t) = B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^4 t}$$

By linearity (or the "principle of superposition"), the general solution to our original problem is:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^4 t}}$$