HW 1 #4

(1) $aw'' + bw' + cw = 0 \qquad a>0,\ b \geq 0,\ c>0$

a) $E(t) = \frac{1}{2}a(w')^2 + \frac{1}{2}cw^2$

$$\begin{aligned} E'(t) &= a w' . w'' + c w w' \\ &= (a w'' + cw) w' \\ &\overset{(1)}{=} -b(w')^2 \leq 0. \end{aligned}$$

So $E'(t) \leq 0$ for all t; in particular, E is nonincreasing $\square$

b) Suppose $w(t)$ solves (1) and $w(0)=0$ and $w'(0)=0$. Then $E(0) = \frac{1}{2}a(w'(0))^2 + \frac{1}{2}c(w(0))^2 = 0$, and since $E(t)$ is nonincreasing (because $E' \leq 0$) and clearly $E(t) \geq 0$, it follows that $E(t) \equiv 0$ for all t. Thus, $w(t) \equiv 0$ for all t. $\square$

c) Suppose $u(t)$ and $v(t)$ solve (1) and $u(0)=v(0)$ and $u'(0)=v'(0)$. By linearity of (1), the function $w(t) := u(t) - v(t)$ is also a solution, and clearly $w(0) = u(0) - v(0) = 0$ and $w'(0) = u'(0) - v'(0) = 0$. Thus, by b), we have that $w(t) \equiv 0$, so $u(t) \equiv v(t)$ for all t.

$\square$